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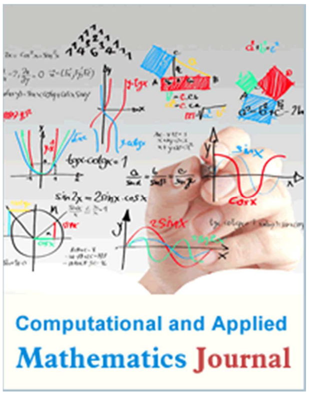

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The fast Fourier transform method for the valuation of European style options in-the-money (ITM), at-the-money (ATM) and out-of-the-money (OTM)

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Abstract
This paper presents the fast Fourier transform method for the valuation of European style options in-the-money, at-the-money and out-of-the-money. The fast Fourier transform method utilizes the characteristic function of the underlying instrument's price process. A fast and accurate numerical solution by means of the Fourier transform method was developed. The fast Fourier transform method is useful for empirical analysis of the underlying asset price. This method can also be used for pricing contingent claims when the characteristic function of the return is known analytically.

1. Introduction

The Black-Scholes model and its extensions constitute the major developments in modern finance. Much of the recent literature on option valuation has successfully applied Fourier analysis to determine option prices such as [1],[3], [5], just to mention a few. These authors numerically solved for the delta and the risk-neutral probability of finishing in-the-money, which can be combined easily with the underlying asset price and the strike price to generate the option value. Unfortunately, this approach is unable to harness the considerable computational power of the fast Fourier transform [7], which represents one of the most fundamental advances in scientific computing. Furthermore, though the decomposition of an option price into probability elements is theoretically attractive as explained in [2], it is numerically undesirable due to discontinuity of the payoffs [4]. For more literature see ([8], [9], [10]) just to mention a few.

2. Fast Fourier Transform Method

The fast Fourier transform is an efficient algorithm for computing the discrete Fourier transform of the form;

$$m(p)=\sum_{j=1}^{N-1}\exp\left(\frac{-2\pi i}{N}pj\right)x_j \tag{1}$$

where N is typically a power of two. (1) reduces the number of multiplications in the required N summations from an order of N^2 to that of $N\ln_2(N)$, a very considerable reduction.

Let p and j be written as binary numbers i.e. $p = 2p_1 + p_0$ and $j = 2j_1 + j_0$ with $j_1, j_0, p_1, p_0 \in \{0,1\}$, then (1) becomes

$$\left.\begin{aligned} m(p_1,p_0) &= \sum_{j_0=0}^{1}\sum_{j_1=0}^{1}\exp\left(\frac{-2\pi i}{N}(2p_1+p_0)(2j_1+j_0)\right)x_{(j_1,j_0)} \\ &= \sum_{j_0=0}^{1}\sum_{j_1=0}^{1}A^{(2p_1+p_0)(2j_1+j_0)}x_{(j_1,j_0)} = \sum_{j_0=0}^{1}\sum_{j_1=0}^{1}A^{2p_0j_1}A^{j_0(2p_1+p_0)}x_{(j_1,j_0)} \end{aligned}\right\} \tag{2}$$

The fast Fourier transform can be described by the following three steps as

$$\left.\begin{aligned} & m^1(p_0,j_0) = \sum_{j_1=0}^{1}x_{(j_0,j_1)}A^{2p_0j_1}, m^2(p_0,j_0) = \sum_{j_1=0}^{1}m^1(p_0,j_0)A^{j_0(2p_1+p_0)}, \\ & m(p_0,p_1) = m^2(p_0,p_1) \end{aligned}\right\} \tag{3}$$

The basic idea of the fast Fourier transform is to develop an analytic expression for the Fourier transform of the option price and to get the price by means of Fourier inversion.

2.1. The Fast Fourier Transform Method for the Valuation of European Call Option

The Fast Fourier transform method is a numerical approach for pricing options which utilizes the characteristic function of the underlying instrument's price process. This approach was introduced by [4]. The Fast Fourier transform method assumes that the characteristic function of the log-price is given analytically.

Consider the valuation of European call option. Let the risk neutral density of $S = \log S_T$ be $f_T(S_T)$. The characteristic function of this density is given by

$$\varphi_T(v) = \int_{-\infty}^{\infty} e^{-2\pi ivS} f(S)dS \tag{4}$$

The price of a European call option with maturity T and exercise price K denoted by $C_T(k)$ is given by

$$C_T(p) = \int_{p}^{\infty}\exp(-rT)(e^S - K)f(S)dS \tag{5}$$

where

$$K = e^p \tag{6}$$

Substituting (6) into (5) yields

$$C_T(p) = \int_{p}^{\infty}\exp(-rT)(e^S - e^p)f(S)dS \tag{7}$$

$$\lim_{k\to-\infty} C_T(p) = \lim_{k\to-\infty}\left(\int_{p}^{\infty}\exp(-rT)(e^S - e^p)f(S)dS\right) = S_0 \tag{8}$$

From (8), it is clearly seen that European call price given by (7) is not square integrable function. We consider a modified version of (7) derived by [4] given by

$$c_T(p) = e^{\alpha p}\widehat{C}_T(p), \quad \alpha > 0 \tag{9}$$

Equation (9) is square integrable in p over the entire real line. Using the definitions of the Fourier transform and its inversion, we have that

$$F(c(p)) = \tilde{c}_T(k) = \int_{-\infty}^{\infty} e^{-2\pi ikp}c_T(p)dp \tag{10}$$

$$F^{-1}(c(p)) = c_T(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{2\pi ikp}\tilde{c}_T(k)dk \tag{11}$$

Substituting (11) into (10) and solving further, then the new call value is obtained as

$$\widehat{C}_T(p) = e^{-\alpha p}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2\pi ikp}\tilde{c}_T(k)dk = e^{-\alpha p}\frac{1}{\pi}\int_{0}^{\infty} e^{-2\pi ikp}\tilde{c}_T(k)dk \tag{12}$$

(13) is a direct Fourier transform and lends itself to an application of fast Fourier transform method. (10) is computed as follows:

$$
\left.\begin{aligned}
\tilde{c}_T(k) &= \int_{-\infty}^{\infty} e^{-2\pi ikp} \int_{p}^{\infty} \exp(-rT)(e^S - e^p) f(S) dS dp \\
&= \int_{-\infty}^{\infty} e^{-rT} f(S) \int_{p}^{\infty} \left(e^{\alpha p}(e^S - e^p)\right) e^{-2\pi ikp} dS dp \\
&= \int_{-\infty}^{\infty} e^{-rT} f(S) \left(\int_{p}^{\infty} \left(e^{\alpha p}(e^S - e^p)\right) e^{-2\pi ikp} dp \right) dS \\
&= \int_{-\infty}^{\infty} e^{-rT} f(S) \left(\frac{e^S}{\alpha + 2\pi ik} \left(e^{(\alpha+2\pi ik)p}\right)\Big|_{-\infty}^{S} - \frac{e^S}{\alpha + 2\pi ik + 1} \left(e^{(\alpha+2\pi ik)p}\right)\Big|_{-\infty}^{S} \right) dS
\end{aligned}\right\} \quad (13)
$$

Taking the limit for $\lim_{p\to-\infty} e^{(\alpha+2\pi ik)p} = 0$ with $\alpha > 0$, then the last equation in (13) becomes

$$
\begin{aligned}
\tilde{c}_T(k) &= \int_{-\infty}^{\infty} e^{-rT} f(S) \left(\frac{\exp(\alpha+1+2\pi ik)S}{\alpha+2\pi ik} - \frac{\exp(\alpha+1+2\pi ik)S}{\alpha+1+2\pi ik} \right) dS \\
&= e^{-rT} \int_{-\infty}^{\infty} f(S) \left(\frac{\exp(\alpha+1+2\pi ik)S}{(\alpha+2\pi ik)(\alpha+1+2\pi ik)} \right) dS \\
&= \frac{e^{-rT}\varphi_T(2\pi k - (\alpha+1)i)}{\alpha^2 + \alpha - 4\pi^2 k^2 + 2\pi ki(2\alpha+1)}
\end{aligned}
$$

Taking the Fourier transform for

$$\int_{-\infty}^{\infty} f(S)\exp\left((\alpha+1+2\pi ik)S\right) = \int_{-\infty}^{\infty} f(S)\exp i\left((k-(\alpha+1)+2\pi k)S\right)$$

we get the characteristic function for the risk neutral price process $\varphi_T(k)(2\pi k - (\alpha+1)i)$.

Finally we have

$$\tilde{c}_T(k) = \frac{e^{-rT}\varphi_T(2\pi k - (\alpha+1)i)}{\alpha^2 + \alpha - 4\pi^2 k^2 + 2\pi k(2\alpha+1)i} \quad (14)$$

Setting $v = 2\pi k$, then (2.57) becomes

$$\tilde{c}_T(v) = \frac{e^{-rT}\varphi_T(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + v(2\alpha+1)i} \quad (15)$$

The corresponding put values can be obtained by defining $p_T(p) = e^{-\alpha p}\widehat{P}_T(p), v = 2\pi k, \alpha > 0$ with Fourier transform

$$\tilde{p}_T(v) = \frac{e^{-rT}\varphi_T(v - (-\alpha+1)i)}{\alpha^2 - \alpha - v^2 + v(-2\alpha+1)i} \quad (16)$$

where φ_T is the Fourier transform of $f(S)$. A sufficient condition for $c_T(p)$ to be square-integrable is given by $\tilde{c}_T(0)$ being finite. This is equivalent to $E^Q(S_T^{\alpha+1}) < \infty$.

Substituting (15) into (13) with $v = 2\pi k$, we have

$$\widehat{C}_T(v) = e^{-\alpha p}\frac{1}{\pi}\int_0^{\infty} e^{-ivp}\left(\frac{e^{-rT}\varphi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+v(2\alpha+1)i}\right)dv \quad (17)$$

Similarly, for the price of put option we have that:

$$\widehat{C}_T(v) = e^{-\alpha p}\frac{1}{\pi}\int_0^{\infty} e^{-ivp}\left(\frac{e^{-rT}\varphi_T(v-(-\alpha+1)i)}{\alpha^2-\alpha-v^2+v(-2\alpha+1)i}\right)dv \quad (18)$$

For the put formula to be well defined, is suffices to choose an appropriate $\alpha > 0$ in a way that $E^Q(S_T^{-\alpha}) < \infty$ (see [6]).

The European call values are calculated using (17). Carr and Madan [4] established that if $\alpha = 0$ the denominator of (17) vanishes when $v = 0$, inducing a singularity in the integrand. Since the fast Fourier transform evaluates the integrand at $v = 0$, the use of the factor $e^{\alpha p}$.

Now, we get the desired option price in terms of $\tilde{c}_T$ using Fourier inversion of the form:

$$\widehat{C}_T(p) = e^{-\alpha p}\frac{1}{\pi}\int_0^{\infty} e^{-ivp}\tilde{c}_T(v)dv \quad (19)$$

Using basic trapezoidal rule, (19) can be computed numerically as:

$$\widehat{C}_T(p) = e^{-\alpha p}\frac{1}{\pi}\sum_{j=0}^{N-1} e^{-ivp}\tilde{c}_T(v_j)\eta \quad (20)$$

where

$$v_j = \eta_j,\ j = 0,1,2,3,...N-1,\ \eta > 0. \quad (21)$$

Since we are interested in (at the money call values) $\widehat{C}_T(p)$, the case where the strikes near the underlying spot price of the asset. This type of options is traded most frequently. The fast Fourier transforms method returns N values of p and we then consider a uniform spacing of size $\rho > 0$ for the log-strikes around the log-spot price S_0 of the form:

$$p = k_u = a + \rho u,\ u = 0,1,2,...,N-1 \quad (22)$$

Equation (22) gives us log-strike levels ranging from –*a* to *a*, where

$$a = -\frac{N\rho}{2} \tag{23}$$

Substituting (22) and (23) into (20), we have

$$\widehat{C}_T(k_u) = \exp\left(\alpha\left(\frac{N\rho}{2} - \rho u\right)\right)\frac{1}{\pi}\sum_{j=0}^{N-1}\exp\left(-iv\left(\frac{-N\rho}{2} + \rho u\right)\right)\tilde{c}_T(v_j)\eta \tag{24}$$

Now, the fast Fourier transforms method can be applied to x_j in (1) provided that $\rho = \frac{2\pi}{\eta N}$. Hence the integration (19) is an application of the summation (1).

For an accurate integration with larger values of η we apply basic Simpson's $\frac{1}{3}$ rule weightings into (24) with the condition $\rho = \frac{2\pi}{\eta N}$, then the accurate call price which is the exact application of the fast Fourier transform method is obtained as:

$$\widehat{C}_T(k_u) = \exp\left(\alpha\left(\frac{N\rho}{2} - \rho u\right)\right)\frac{1}{\pi}\sum_{j=0}^{N-1}\exp\left(\frac{-2\pi i}{N}u + iav_j\right)\tilde{c}_T(v_j)\frac{\eta}{3}\left(3 + (-1)^j - \delta_{j-1}\right) \tag{25}$$

where $k_u = a + \rho u, u = 0,1,2,...,N-1$ and δ_j is called the Kronecker delta function expressed as:

$$\delta_j = \begin{cases} 0, & if\ j \neq 0 \\ 1, & if\ j = 0 \end{cases} \tag{26}$$

2.2. The Fast Fourier Transform for the Valuation of Options in-the-Money (ITM), at-the-Money (ATM) and out-of-the-Money (OTM)

For in-the-money and at-the-money options prices, call values are calculated by an exponential function to obtain square integrable function whose Fourier transform is an analytic function of the characteristic function of the log-price. Unfortunately, for very short maturities, the call value approaches to its non-analytic intrinsic value causing the integrand in the inversion formula of Fourier transforms to vary above and below a mean value and therefore remains tedious to be integrated numerically. We introduce an alternative approach called the "Time Value Method" proposed by [4] to mitigate this numerical inconvenience. This approach works with time values only, which is quite similar to the previous approach. But in this context the call price is obtained by means of the Fourier transform of a modified time value, where the modification involves a hyperbolic sine function instead of exponential function.

Let $z_T(k)$ represent the time value of out-of -the-money (OTM), that is, for $S < k$ we have the call price for $z_T(k)$ and for $S > k$ we have the put option for $z_T(k)$. Scaling $S_0 = 1$ for simplicity, $z_T(k)$ is defined by

$$z_T(k) = e^{-rT}\int_{-\infty}^{\infty}\left(e^S - e^k\right)\mathrm{I}_{k>S,k<0}f(S)dS + e^{-rT}\int_{-\infty}^{\infty}\left(e^S - e^k\right)\mathrm{I}_{k<S,k>0}f(S)dS \tag{27}$$

Where

$f(S)$ is the risk-neutral density of the log-price S and I denotes the indicator function. Let the Fourier transform of $z_T(k)$ be defined by

$$\upsilon_T(v) = \int_{-\infty}^{\infty} e^{ivk} z_T(k)dk \tag{28}$$

The prices of OTM options can be obtained by the inversion formula of the Fourier transform of (28) of the form

$$z_T(k) = \int_{-\infty}^{\infty} e^{-ivk}\upsilon_T(v)dv \tag{29}$$

By substituting (27) into (228) and writing in terms of characteristic functions then (28) becomes

$$\left.\begin{aligned}\upsilon_T(v) &= \int_{-\infty}^{\infty} e^{ivk}\left(e^{-rT}\int_{-\infty}^{\infty}\left(e^S - e^k\right)\mathrm{I}_{k>S,k<0}f(S)dS + e^{-rT}\int_{-\infty}^{\infty}\left(e^S - e^k\right)\mathrm{I}_{k<S,k>0}f(S)dS\right)dk \\ &= \frac{e^{-rT}}{1+iv} - \frac{e^{-rT}\varphi_T(-i)}{iv} - \frac{e^{-rT}\varphi_T(v-i)}{iv(1+iv)}\end{aligned}\right\} \tag{30}$$

There are no issues regarding the integral of this function in (29) as $k \to -\infty \; or \; \infty$, the time value at k tends to zero can get rather steep as T tends to zero and this can make the Fourier transform to be wide and oscillatory. By considering the damping function $\sinh(\alpha k)$, the time value follows a Fourier inversion:

$$z_T(k) = \frac{1}{\pi \sinh(\alpha k)} \int_{-\infty}^{\infty} e^{-ivk} \zeta_T(v) dv \qquad (31)$$

where

$$\zeta_T(v) = \int_{-\infty}^{\infty} e^{ivk} \sinh(\alpha k) z_T(k) dk \qquad (32)$$

Solving (2.72) further and replace $\sinh(\alpha k)$ by $\frac{e^{\alpha k} - e^{-\alpha k}}{2}$, then we have

$$\left.\begin{aligned} \zeta_T(v) &= \int_{-\infty}^{\infty} e^{ivk} \left(\frac{e^{\alpha k} - e^{-\alpha k}}{2} \right) z_T(k) dk \\ &= -\left(\frac{\upsilon_T(v+i\alpha)}{2} - \frac{\upsilon_T(v-i\alpha)}{2} \right) \end{aligned}\right\} \qquad (33)$$

The use of the fast Fourier transform for calculating out-of-the-money option prices is similar to (25). The only differences are that they replace the multiplication by $\exp(-\alpha k_u) = \exp(-\alpha(a+\rho u)) = \exp\left(\alpha\left(\frac{N\rho}{2} - \rho u\right)\right)$ with a division by $\sinh(\alpha k)$ and the function call to $\tilde{c}_T(v_j)$ be replaced by a function $\zeta_T(v)$. Hence the formula for out-of-the-money option price is given by

$$\hat{C}_T(k_u) = \exp(-\alpha k_u) \frac{1}{\pi \sinh(\alpha k_u)} \sum_{j=0}^{N-1} \exp\left(\frac{-2\pi i}{N} u + iav_j \right) \zeta_T(v_j) \frac{\eta}{3} \left(3 + (-1)^j - \delta_{j-1} \right) \qquad (34)$$

where $k_u = a + \rho u = \left(\frac{N\rho}{2} - \rho u \right), u = 0, 1, 2, ..., N-1$ and δ_j is called the Kronecker delta as given by (26).

3. Numerical Experiment and Discussion of Results

This section presents numerical experiment and discussion of results as follows:

Numerical Example

We consider the pricing of "in-the-money (ITM) and at-the-money (ATM)" and "out-of-the-money (OTM)" European call option with the following parameters given below:

$$\left.\begin{aligned} &\kappa = 0.8, \\ &\theta = 0.5, \\ &\sigma = 0.5, \\ &v_0 = 0.8, \\ &\rho = \{-0.5, 0, 0.5\}, \\ &S_0 = 60, \\ &r = 0.08, \\ &T = 0.75, \\ &K = \{20, 40, 60, 80, 100\} \end{aligned}\right\} \qquad (35)$$

We examine the effects of correlation coefficient, strike price and volatility of volatility; on option values using the fast Fourier transform method. The results generated are shown in the Tables 1A and 1B for "in-the-money (ITM) and at-the-money (ATM)" and "out-of-the-money (OTM)" European call options respectively.

Table 1A. *The Effects of Correlation ρ and Exercise Price K on "in-the-money (ITM) and at-the-money (ATM) European Options Values"*

K	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$
20	41.7372	41.4985	41.1620
40	27.7962	27.6276	27.4127
60	18.2136	18.4365	18.6684
80	11.9475	12.5720	13.1965
100	7.8495	8.7680	9.6557

Table 1B. *The Effects of Correlation ρ, and Exercise Price K on "out-of-the-money (OTM) European Options Values"*

K	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$
20	41.9377	41.7842	41.6060
40	27.8097	27.6654	27.4978
60	18.1999	18.4337	18.6880
80	11.9280	12.5585	13.1959
100	7.8289	8.7510	9.6468

3.1. Discussion of Results

Tables 1A and 1B show the effect of correlation coefficient ρ and strike price K on ITM, ATM and OTM options values using fast Fourier transform. The price of option and the time value for Nine-month call options associated with volatility of volatility $\sigma = 0.5$ are relatively close as we can see from the Tables 1A and 1B for ITM, ATM and OTM respectively. The effect of correlation coefficient depends on the relationship between the current underlying price of the asset and strike price. For a positive correlation coefficient, the price of out-the-money call option becomes lower. For negative correlation coefficient, the price of in-the-money and at-the-money call options becomes higher. When the correlation coefficient is zero, the effect of volatility of volatility is negligible.

The fast Fourier transform method is useful for empirical analysis of the underlying asset price. This method can also be used for pricing contingent claims when the characteristic function of the return is known analytically.

4. Conclusion

The fast Fourier transform method is used because of its advantages when compared to the analytic solution. Using the fast Fourier transform with risk neutral approach provides simplicity in calculations.An analysis of the fast Fourier transform method reveals that the volatility of volatility σ and the correlation coefficient ρ have significant impact on option values, especially long-time option, stock returns and negatively correlated with volatility and these negative correlations are important for the valuation of European style options.

References

[1] Bakshi, G., and Chen, Z. (1997), An alternative valuation model for contingent claims.*Journal of Financial Economics*, 44(1), 123—165

[2] Bakshi, G., and Madan, D.B. (1999), Spanning and derivative security valuation.*Journal of Financial Economics*,

[3] Bates, D. (1996). Jumps and stochastic volatility: Exchange rate processes implicit inDeutschemark options, *Review of Financial Studies*, 9, 69--108.

[4] Carr, P. and Madan, D. (1999), Option valuation using the fast Fourier trans-form, *Journal of Computational Finance*, 2, 61-73.

[5] Chen, R.R., and Scott, L. (1992), Pricing interest rate options in a two-factor Cox-Ingersoll-Ross model of the term structure, *Review of Financial Studies*, 5, 613--636.

[6] Lee, R., (2004), Option pricing by transform methods: extensions, unificationand error control, *Journal of Computational Finance*, 7.

[7] Walker, J.S. (1996). Fast Fourier Transforms, CRC Press, Boca Raton, Florida.

[8] Wu, L. (2008), Modeling financial security returns using Levy processes, In: Handbooks in operations research and management science: Financial Engineering, Volume 15, Eds. J. Birge and V. Linetsky, Elsevier, North-Holland.

[9] Yan, G. and Hanson F. B. (2006), Option pricing for a stochastic volatility jump-Diffusion Model with Log-Uniform jump-amplitudes, Proceedings of the 2006 American control conference, Minneapolis, Minnesota, June 14–16.

[10] Zeliade systems (2009). Heston 2009, Zeliade systems white paper, September. Available at http://www.zeliade.com/whitepapers/zwp-0004.pdf.