The inverse eigenvalue problem for some special kind of matrices (I)

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Citation

Abstract
In recent paper [1] (Juang Peng, Xi-Yan Hu, Lei Zhang) two inverse eigenvalue problems are solved and in the order article [2] (Hubert Pickmann, Juan Egana, Ricardo L. Soto), a correction, for one of the problems stated in the first article, has been presented as well. In this article, according to the article [2], a solution which is different from the one in the article [1] has been presented for one of the problems which are in the article [1]. The matrix solution in the article [1] and the one which is presented by us, in the main diagonal, are similar, but instead of first column and row, we valued second column and row, furthermore other element of the matrix are considered null.

1. Introduction

In recent paper [1], an inverse eigenvalue problem is solved, a part of which, considering

\[ \lambda^{(n)}_1 < \lambda^{(n-1)}_1 < \cdots < \lambda^{(2)}_1 < \lambda^{(1)}_1 < \lambda^{(2)}_1 < \cdots < \lambda^{(n)}_n, \]

finds an \( n \times n \) matrix \( B_n \), such that \( \lambda^{(j)}_1 \) and \( \lambda^{(j)}_1 \) are the minimal and maximal eigenvalues of \( B_j \) respectively for all \( j = 1, 2, 3, \ldots, n \), where \( B_j \) to denote the \( j \times j \) leading principal submatrix of \( B_n \), in which \( B_n \) is as below:

\[
B_n = \begin{pmatrix}
    a_1 & b_1 & b_2 & b_3 & \cdots & b_{n-1} \\
    b_1 & a_2 & 0 & 0 & \cdots & 0 \\
    b_2 & 0 & a_3 & 0 & \cdots & 0 \\
    b_3 & 0 & 0 & a_4 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    b_{n-1} & 0 & 0 & 0 & \cdots & a_n
\end{pmatrix}
\]

where \( a_i \) are distinct for all \( i = 1, 2, 3, \ldots, n \) and all \( b_i \) are positive. Then consider the following matrix:

\[
A_n = \begin{pmatrix}
    a_1 & b_1 & 0 & 0 & \cdots & 0 \\
    b_1 & a_2 & b_2 & b_3 & \cdots & b_{n-1} \\
    0 & b_2 & a_3 & 0 & \cdots & 0 \\
    0 & b_3 & 0 & a_4 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & b_{n-1} & 0 & 0 & \cdots & a_n
\end{pmatrix}
\]  \hspace{1cm} (1)

where \( a_i \) are distinct for all \( i = 1, 2, \ldots, n \) and all \( b_i \) are positive. Throughout this paper, we use \( A_n \) to denote a special kind of matrices defined as in (1) and \( A_j \) to denote the \( j \times j \) leading principal submatrix of \( A_n \).
In this paper, we, like paper [1], construct a matrix $A_n$ under the following condition:

For $2n - 1$ given real numbers $\lambda^{(n)}_1 < \lambda^{(n-1)}_1 < \ldots < \lambda^{(2)}_1 < \lambda^{(1)}_2 < \lambda^{(2)}_2 < \ldots < \lambda^{(n)}_n$, we find an an $n \times n$ matrix $A_n$, such that $\lambda^{(j)}_1$ and $\lambda^{(j)}_n$ are the minimal and maximal eigenvalues of $A_j$ respectively for all $j = 1, 2, 3, \ldots, n$.

One of the main problems of the theory of matrices is inverse eigenvalue problem that has some properties of a Sturm sequence, satisfying the following recurrence relation

\[ \text{Coefficient of the largest power of } x \text{ is positive then for } x < x^{(i-1)} \text{ the polynomial } \varphi_{i-1}(x) \text{ near } -\infty \text{ Equals } (-1)^{i+1} \text{ or } (-1)^{i-1} \text{ then table of sign of } \varphi_{i-1}(x) \text{ is:} \]

\[
\begin{array}{cccccc}
 x & -\infty & x^{(i-1)} & \ldots & x^{(i)} & \infty \\
 \varphi_{i-1}(x) & (-1)^{i+1} & 0 & \ldots & (-1)^{i+k} & 0 \\
 \end{array}
\]

If $x^{(i)}_k$ is a root of $\varphi_i(x)$ then

\[ 0 = (x^{(i)}_k - a_i)\varphi_{i-1}(x^{(i)}_k) - b^2_{i-1}(x^{(i)}_k - a_i) \cdots (x^{(i)}_k - a_{i-1}) \] (3)

And

\[ \varphi_{i+1}(x^{(i)}_k) = (x^{(i)}_k - a_{i+1})\varphi_i(x^{(i)}_k) - b^2_i(x^{(i)}_k - a_i) \cdots (x^{(i)}_k - a_i) \] (4)

From (3) and (4) we have

\[ \varphi_{i-1}(x^{(i)}_k) = \frac{b^2_{i-1}(x^{(i)}_k - a_i) \cdots (x^{(i)}_k - a_{i-1})}{x^{(i)}_k - a_i} \times \frac{b^2_i(x^{(i)}_k - a_i)}{(x^{(i)}_k - a_i) b^2_i} = \frac{-\varphi_{i+1}(x^{(i)}_k) b^2_{i-1}}{(x^{(i)}_k - a_i)^2 b^2_i} \]

consequently $\varphi_{i-1}(x^{(i)}_k)$ and $\varphi_{i+1}(x^{(i)}_k)$ have opposite signs.

Now since $\varphi_{i-1}$ and $\varphi_{i+1}$ have opposite signs, the table of $\varphi_{i+1}(x)$ sign in root of $\varphi_i(x)$ is:

\[
\begin{array}{cccccc}
 x & -\infty & x^{(i)}_1 & x^{(i)}_2 & \ldots & x^{(i)}_l & \infty \\
 \varphi_{i+1}(x) & (-1)^{i+1} & (-1)^{i+2} & (-1)^{i+3} & \ldots & (-1)^{2i+1} & (-1)^{2i+2} \\
 \end{array}
\]
and notice that the sign of $\varphi_{i+1}(x)$ in interval $(-\infty, x_1^{(i)})$ and $(x_1^{(i)}, \infty)$ changes and by Bolzano-Weierstrass theorem, $\varphi_{i+1}(x)$ in both interval has root.

Then we have

$$x^{(i+1)} - x_1^{(i)} < \cdots < x^{(i)} - x_1^{(i)}.$$ 

According what we mentioned above all $\varphi_i(x)$ has simple roots and since in $i$ intervals the sign of each $\varphi_i$ changes and also have $i$ roots, then all roots are real. Since $\varphi_0 = 1$, considering what we said above \{\varphi_i\} have some properties of a Sturm sequence.

**Lemma 3.** Assume

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_n^{(n)},$$

be real numbers, where $\lambda_1^{(i)}$ and $\lambda_2^{(i)}$ are the minimal and maximal eigenvalues of $A_j$, then we have

$$\lambda_1^{(i)} < a_i < \lambda_j^{(i)}, \text{for } j = 2, \ldots, n, i = 1, 2, \ldots, j$$

**Proof.** By Induction on, if $j = 2$, since $\varphi_2(\lambda) = (\lambda - a_1)(\lambda - a_2) - b_1^2$, then $\varphi_2(a_1) = \varphi_2(a_2) = -b_1^2 < 0$, with respect to sign of $\varphi_2(\lambda)$ between $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ we have

$$\lambda_1^{(2)} < a_1, a_2 < \lambda_2^{(2)}$$

Now let $\lambda_1^{(j-1)} < a_i < \lambda_1^{(j-1)}$, for $j = 2, \ldots, j-1$, we show that $\lambda_1^{(j)} < a_i < \lambda_1^{(j)}$, $i = 1, \ldots, j$ since for $i = 1, \ldots, j-1$ we have

$$\lambda_1^{(j)} < \lambda_1^{(j-1)} < a_i < \lambda_1^{(j-1)} < \lambda_1^{(j)}$$

It is enough to prove that $\lambda_1^{(j)} < a_j < \lambda_1^{(j)}$. The sign of $\varphi_j(\lambda)$ in interval $(-\infty, \lambda_1^{(j)})$ is $(-1)^j$ and in $(\lambda_1^{(j)}, \lambda_2^{(j)})$ will be $(-1)^{j-1}$ and $\varphi_j(\lambda)$ has negative sign in $(\lambda_1^{(j)}, \lambda_2^{(j)})$ and positive sign in $(\lambda_1^{(j)}, +\infty)$.

Applying lemma 1, we have:

$$0 = \varphi_j(\lambda_1^{(j)}) = (\lambda_1^{(j)} - a_j) \varphi_{j-1}(\lambda_1^{(j)}) - b_{j-1}^2 (\lambda_1^{(j)} - a_{j-1}) (\lambda_1^{(j)} - a_{j-2}) \cdots (\lambda_1^{(j)} - a_1)$$

In (5), the right hand side expression has sign $(-1)^{j-1}$ in which the first expression $\varphi_{j-1}(\lambda_1^{(j)})$ has sign $(-1)^{j-1}$, if $(-1)^{j-1}$ is root of $\varphi_j(\lambda)$, this will be necessary for $(\lambda_1^{(j)}, a_j)$ to have negative sign. This means

$$\lambda_1^{(j)} < a_j$$

if we repeat this reason for $\varphi_j(\lambda_1^{(j)})$, then we will have $a_j < \lambda_1^{(j)}$, and this proves our claim.

**Corollary 1.** If $\lambda_1^{(j)}$ and $\lambda_2^{(j)}$ are the minimal and maximal zeros of $\varphi_j(\lambda)$ respectively, then

1) for $\mu < \lambda_1^{(j)}$, we have $(-1)^j \varphi_j(\mu) > 0$, 
2) for $\mu > \lambda_1^{(j)}$, we have $\varphi_j(\mu) > 0$, $j = 1, 2, \ldots, n$.

### 3. Existence and Uniqueness

**Theorem 1.** (existence and uniqueness of matrix $A_n$)

Let $\lambda_1^{(i)}$ and $\lambda_2^{(i)}$ for $j = 1, 2, \ldots, n$ be real and satisfy in the following relation:

$$\lambda_1^{(1)} < \lambda_1^{(2)} < \lambda_1^{(3)} < \cdots < \lambda_1^{(n)}$$

Then there exist the unique matrix $A_n$ in form (4) with $a_i \neq a_j (i = 1, 2, \ldots, n)$ and $b_1 > 0$, where $\lambda_1^{(i)}$ and $\lambda_2^{(i)}$ are minimal and maximal eigenvalues problem of $A_j$ respectively.

If

$$\lambda_1^{(2)} + \lambda_2^{(2)} \neq 2 \lambda_1^{(1)}$$

and

$$\frac{\lambda_1^{(j-1)} - \lambda_1^{(j)}}{\lambda_1^{(j-1)} - \lambda_1^{(j-1)}} \leq \frac{\varphi_{j-1}(\lambda_1^{(j)}) \Pi_{i=1}^{j-1} (\lambda_1^{(j)} - a_i)}{\varphi_{j-1}(\lambda_1^{(j)}) \Pi_{i=1}^{j-1} (\lambda_1^{(j)} - a_i)}$$

for $j = 3, 4, \ldots, n$ or

$$\frac{\lambda_1^{(j-1)} - \lambda_1^{(j)}}{\lambda_1^{(j-1)} - \lambda_1^{(j-1)}} \geq \frac{\varphi_{j-1}(\lambda_1^{(j)}) \Pi_{i=1}^{j-1} (\lambda_1^{(j)} - a_i)}{\varphi_{j-1}(\lambda_1^{(j)}) \Pi_{i=1}^{j-1} (\lambda_1^{(j)} - a_i)}$$

for $j = 3, 4, \ldots, n$, and if we define

$$u_j = \varphi_{j-1} \left( \lambda_1^{(j)} \right) \Pi_{i=1}^{j-1} (\lambda_1^{(j)} - a_i)$$

and

$$v_j = \varphi_{j-1} \left( \lambda_1^{(j)} \right) \Pi_{i=1}^{j-1} (\lambda_1^{(j)} - a_i)$$

for $j = 3, 4, \ldots, n$, and

$$h_j = \varphi_{j-1} \left( \lambda_1^{(j)} \right) \Pi_{i=1}^{j-1} (\lambda_1^{(j)} - a_i) - \varphi_{j-1} \left( \lambda_1^{(j)} \right) \Pi_{i=1}^{j-1} (\lambda_1^{(j)} - a_i)$$

Then we can find $a_i$, $b_j$ by the following relations:
\[ a_1 = \lambda_1^{(1)}, \quad a_2 = \lambda_2^{(2)} + \lambda_1^{(2)} - \lambda_1^{(1)}, \quad b_1^2 = \left( \lambda_1^{(2)} - \lambda_1^{(1)} \right) \left( \lambda_1^{(2)} - \lambda_2^{(2)} \right), \]

\[ a_j = \frac{\lambda_j^{(j-1)} u_j - \lambda_j^{(j)} v_j}{h_j}, \quad j = 3, 4, ..., n \]

(9)

\[ b_{j-1}^2 = \frac{\lambda_j^{(j-1)} u_j - \lambda_j^{(j)} v_j}{h_j}, \quad j = 3, 4, ..., n \]

(10)

**Proof:** At first we prove that \( a_j \) and \( b_{j-1} \) exist for all \( j \), if we denote:

\( h_j \) is the denominator of \( a_j \) and \( b_{j-1} \), and we prove that it is always nonzero. The sign of \( \varphi_{j-1} \left( \lambda_j^{(j-1)} \right) \prod_{i=1, i \neq 2}^{j-1} \left( \lambda_i^{(j)} - \lambda_i^{(1)} \right) \) is \((-1)^{j-1}\) and since \( a_i < \lambda_j^{(j)} \) for \( i = 1, 2, ..., j \), then

\[ \prod_{i=1, i \neq 2}^{j-1} \left( \lambda_i^{(j)} - \lambda_i^{(1)} \right) > 0 \quad \text{and the sign of} \quad \varphi_{j-1} \left( \lambda_j^{(j-1)} \right) \]

is \((-1)^{j-1}\) and it is nonzero, then denominator of both terms with same sign and nonzero, is nonzero. Then \( a_j, b_{j-1}^2 \) exist. Furthermore

\[ b_{j-1}^2 = \frac{\lambda_j^{(j-1)} u_j - \lambda_j^{(j)} v_j}{h_j} \]

(11)

In numerator of (11) sign of \( \left( \lambda_j^{(j)} - \lambda_j^{(1)} \right) \) and \( \varphi_{j-1} \left( \lambda_j^{(1)} \right) \) is positive and \( \varphi_{j-1} \left( \lambda_j^{(j)} \right) \) has sign \((-1)^{j-1}\), then the sign of numerator is \((-1)^{j-1}\) and the denominator of this rational expression has sign \((-1)^{j-1}\), therefore \( b_{j-1}^2 \) is positive.

Now we prove that \( a_i \) which attained are distinct. From \( \lambda_1^{(2)} + \lambda_2^{(3)} = 2 \lambda_1^{(1)} \), we have \( \lambda_1^{(2)} + \lambda_2^{(3)} - \lambda_1^{(1)} \neq \lambda_2^{(1)} \) consequently \( a_2 \neq a_1 \).

Let

Now we explain \( j = 3 \):

The relation (7) includes

\[ \frac{\lambda_2^{(2)} - \lambda_3^{(3)}}{\lambda_2^{(2)} - \lambda_1^{(3)}} < \frac{(-1)^2 u_3}{(-1)^2 v_3} \]

since \( v_3 \) is negative then \((-1)^2 (\lambda_1^{(3)} u_3 - \lambda_2^{(3)} v_3) > \lambda_2^{(2)} (-u_3 - v_3) \), finally

\[ a_3 = \frac{(-1)^2 (\lambda_1^{(3)} u_3 - \lambda_2^{(3)} v_3)}{(-1)^2 (u_3 - v_3)} > \lambda_2^{(2)} \]

whereas \( \lambda_3^{(2)} < a_1, a_3 < \lambda_2^{(2)} \), then \( a_3 \neq a_1 \neq a_2 \).

Now we assume \( a_i \) for \( i = 1, 2, ..., j - 1 \) are distinct, by relation (7) we have

\[ \frac{\lambda_j^{(j-1)} - \lambda_j^{(j-1)}}{\lambda_j^{(j-1)} - \lambda_1^{(j-1)}} < \frac{(-1)^{j-1} u_j}{(-1)^{j-1} v_j} \]

note that \((-1)^{j-1} v_j \) is negative, then

\[ \frac{(-1)^{j-1} (u_j - \lambda_j^{(j)} v_j)}{(-1)^{j-1} (u_j - v_j)} > \lambda_1^{(j-1)} \]

this means \( a_j > \lambda_1^{(j)} \) and since \( \lambda_1^{(j-1)} < a_i < \lambda_1^{(j)} \) for \( i = 1, ..., j - 1 \), then we have

\[ a_j \neq a_j^{-1} \neq \cdots \neq a_1. \]

If we use relation (8) we conclude that \( a_j < \lambda_1^{(j-1)} \), in which we take distinct \( a_i \) for \( i = 1, ..., j \), then the problem has solution and equivalently the following equations:

\[ \varphi_j \left( \lambda_j^{(j)} \right) = 0, \quad \varphi_j \left( \lambda_j^{(1)} \right) = 0 \]

which has solution distinct \( a_i \) for all \( i = 1, ..., n \) and \( b_{j-1} \) satisfying \( b_j \) for all \( j = 2, ..., n \). If problem has solution, then

\[ \varphi_j \left( \lambda_j^{(1)} \right) = 0 \Rightarrow a_1 = \lambda_j^{(1)} \]

\[ \varphi_j \left( \lambda_j^{(2)} \right) = (\lambda_j^{(2)} - a_2) (\lambda_j^{(2)} - a_2) - b_1^2 = 0 \]

\[ \varphi_j \left( \lambda_j^{(3)} \right) = (\lambda_j^{(3)} - a_3) (\lambda_j^{(3)} - a_3) - b_1^2 = 0 \]

(12)

then the simplifying get:

\[ a_2 = \frac{\lambda_2^{(2)} - \lambda_1^{(1)}}{\lambda_2^{(3)} - \lambda_1^{(2)}} \]

with substituting \( a_2 \) in (12) we have

\[ b_2^2 = \left( \lambda_2^{(2)} - \lambda_1^{(1)} \right) \left( \lambda_1^{(3)} - \lambda_2^{(2)} \right) \]

and

\[ \varphi_j \left( \lambda_j^{(1)} \right) = \left( \lambda_j^{(1)} - a_1 \right) \varphi_j - \left( \lambda_j^{(1)} - b_j \right) = 0 \]

\[ \varphi_j \left( \lambda_j^{(2)} \right) = \left( \lambda_j^{(2)} - a_2 \right) \varphi_j - \left( \lambda_j^{(2)} - b_j \right) = 0 \]

(13)

for \( 3 < j < n \) note that with

\[ (13) \times \prod_{i=1, i \neq 2}^{j-1} \left( \lambda_j^{(j)} - a_i \right) - (14) \times \prod_{i=1, i \neq 2}^{j-1} \left( \lambda_j^{(1)} - a_i \right) = 0 \]

we find (9) and with

\[ (13) \times \varphi_j \left( \lambda_j^{(1)} \right) - (14) \times \varphi_j \left( \lambda_j^{(1)} \right) \]

we find (10). Finally uniqueness matrix \( A_n \) by (9) and (10) is trivial.

**Example 1**

For given 7 real numbers

\[ a_1^{(1)} = 6, \lambda_2^{(1)} = 4, \lambda_2^{(2)} = 7, \lambda_3^{(2)} = 3, \lambda_3^{(3)} = 9, \lambda_4^{(4)} = 1, \lambda_4^{(4)} = 14, \]
Finding a 4 × 4 matrix $A_4$ that $\lambda^{(j)}_1$ and $\lambda^{(j)}_2$ are the minimal and maximal eigenvalue of its $j \times j$ leading principal submatrix. By applying theorem 1, we get the unique solution

$$A_4 = \begin{pmatrix} 6 & \sqrt{2} & 0 & 0 \\ \sqrt{2} & 5 & 2.390457218 & 5.108115212 \\ 0 & 2.390457218 & 7.285714286 & 0 \\ 0 & 5.108115212 & 0 & 10.69666390 \end{pmatrix}$$

From the above matrix $A_4$ we compute the spectrum of $A_4$, and get

$$\lambda(A_1) = 6, \lambda(A_2) = 4.7,$$

$$\lambda(A_3) = 2.99999990, 6.285714319, 8.999999981,$$

$$\lambda(A_4) = 1.00000016, 6.166248253, 7.816129917, 14.00000002.$$

those obtained data show that Algorithm is correct.

References


