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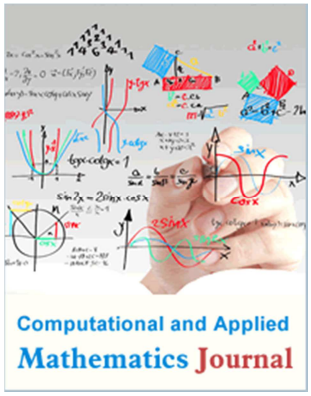

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Numerical Solution of Boundary Value Problems for the Laplacian in R^3 in the Case of Complex Boundary Surface

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Abstract
Potential theory is one of the ways to solve the boundary value problems for the Laplacian. Well-posed solvability of integral equations equivalent to boundary value problems allow to use for their solution well known projection methods. In many applied problems the boundary surfaces have complex geometry and contain the edges and corner points. Together with the singularity in the kernel this gives rise to a singularity in the searched density of the potential. The methods are proposed for removal of singularities in the kernels and unknown densities of potentials that significantly improve the accuracy of projection methods, as well as their numerical solution.

1. Introduction

During modeling of many physical processes (diffusion, heat flow, electrostatic field, frictionless flow, elastic motion of solids etc.), the need for solution of boundary value problems for the Laplacian in R^3 arises [1]. In the case of complex geometry of the boundary surface for the solution of such problems should be used the potential theory methods [2]. Depending on the properties of environment solution of boundary value problem can be sought in the form of simple layer potential [3], double layer potential [4], or the sum of these potentials [5]. Systems of integral equations equivalent to the boundary value problems may contain integral equations of the first kind [2-5]. Review of the conditions of well-posed solvability of these equations for the simple and double layer potentials and closed boundary surfaces contained in [6], for tired boundary surfaces – in [7]. Review of the conditions of well-posed solvability of these equations for the sum of potentials contained in [8]. These results allow to use for the the solution of systems of integral equations the well-known projection methods [2, 9-13]. Implementation of these methods requires calculation of singularities in kernel and unknown density of potentials when approaching the edge or the corner of the boundary surface, which often occurs in existing devices, such as high voltage transmission towers, radar systems of various types etc. This problem was studied in [14-19]. In the given articles, analytical methods for calculation of singularities for particular surface are proposed. The use of finite element approximation of unknown potential density provides additional opportunities to calculate its singularities and build new algorithms for calculation of matrix coefficients of the system of linear algebraic equations, discretized system integral equations, and significantly improves the accuracy of their numerical solution.

2. Collocation Method and Function of Singularity of Potential Density

Let $\Gamma = \bigcup_{i=1}^{N} \Gamma_i$ be the boundary surface in R^3. Suppose that on each surface Γ_i is given boundary condition $\gamma_{j,i} u(x) = f_{j,i}(x)$, $x \in \Gamma_i$, $i = \overline{1,N}$, $j = \overline{1,M}$, which must be satisfied the searched harmonic function $u(x)$, $x \in R^3$. Trace operator $\gamma_{j,i}$ of function $u(x)$ on surface Γ_i determines [20] the type of the boundary condition (Dirichlet, Neumann, Poincare, jump unknown function or (and) its normal derivative, etc.) and $f_{j,i}$ sets the value of this condition. The boundary conditions can be as one-sided and double-sided [6-8]. They may also be mixed when on different sides of the boundary surface set the boundary conditions of different types [5]. Denote

$$u(x) = W(\sigma, q)(x) = (U, V)(\sigma, q)^T(x),$$

where

$$(U\sigma)(x) = \frac{1}{4\pi} \int_{\Gamma} \sigma(y) G(x,y) d\Gamma_y,$$

$G(x,y) = \frac{1}{|x-y|}$, $x \in R^3$, is a simple layer potential and $\sigma(y)$, $y \in \Gamma$, is a density of simple layer potential, and

$$(Vq)(x) = -\frac{1}{4\pi} \int_{\Gamma} q(y) Q(x,y) d\Gamma_y,$$

$Q(x,y) = \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|}$, $x \in R^3$, is a double layer potential and $q(y)$, $y \in \Gamma$, is a density of double layer potential. To determine the unknown densities of the potentials need to solve the system of integral equations

$$\sum_{k=1}^{N} \gamma_{j,k} W(\sigma, q)(x) = f_{j,i}(x), x \in \Gamma_i, \quad (1)$$

$$i = \overline{1,N}, \ j = \overline{1,M}.$$

Well-known projection methods (collocation, Galerkin, least squares, etc.) can be used to solve the system (1). These methods are well investigated for stability and convergence and commonly used to solve many problems of mathematical physics [2-4, 13, 14].

Let $S_i = [0, a_i] \times [0, b_i] \in R^2$ be the rectangular domain of parameters of surface Γ_i in which a rectangular grid $S_{h,i}$ with steps $h_{\tau,i} = a_i / N_{\tau,i}$ and $h_{\nu,i} = b_i / N_{\nu,i}$, $i = \overline{1,N}$, is generated. Let $\Phi_{N_{\tau,i}, N_{\nu,i}}(\tau, \nu) = = \{\varphi_{lm}(\tau, \nu)\}_{l=0,\, m=0}^{N_{\tau,i}\ N_{\nu,i}}$, $\tau \in [0, a_i]$, $\nu \in [0, b_i]$, be the system of piecewise-linear finite elements defined on $S_{h,i}$. Consider collocation method for solution of the system (1) when the set of points of observation coincides with the nodes of the grid $S_{h,i}$. System of collocation equations in this case has the form

$$\sum_{k=1}^{N} \gamma_{j,k} W\Big(\sum_{l_k=0}^{N_{\tau,k}} \sum_{m_k=0}^{N_{\nu,k}} \phi_{lm}(\tau, \nu), \sum_{\tilde{l}_k=0}^{N_{\tau,k}} \sum_{\tilde{m}_k=0}^{N_{\nu,k}} \phi_{\tilde{l}\tilde{m}}(\tau, \nu)\Big)(x_{p_i,t_i}) = f_{j,i}(x_{p_i,t_i}), x_{p_i,t_i} \in \Gamma_i \quad (2)$$

$$p_i = \overline{0(1)N_{\tau,i}}, \ t_i = \overline{0(1)N_{\nu,i}}, \ i = \overline{1,N}, \ j = \overline{1,M}.$$

where a_{lm}, $b_{\tilde{l}\tilde{m}}$ are unknown parameters and $x_{p_i,t_i} \in \Gamma_i$, $i = \overline{1,N}$, is the set of points of observation. Conditions of stability and convergence of the method (2) defined in [13].

The main factor affecting the accuracy of approximate solution of the system of collocation equations is the calculation of integrals with singularities in their matrices coefficients. Let $P = [\tau_0, \tau_1] \times [\nu_0, \nu_1]$ be the certain element of the grid $S_{h,i}$, $\tau_1 - \tau_0 = h_{\tau,i}$, $\nu_1 - \nu_0 = h_{\nu,i}$, $i = \overline{1,N}$. Elimination of singularity of the density is only suitable for grid elements that are tangent to the edges or corners of the boundary surface. Moreover, construction of the function of singularity for domain of parameters S_i is often a difficult problem, e.g., for surfaces with curved cuts. Consider the elements of the matrix of the system of collocation equations that are formed from the integrals depending on the given boundary conditions:

$$I_1 = \int_{\tau_0}^{\tau_1} \int_{\nu_0}^{\nu_1} \frac{\rho(\tau,\nu) L(\tau,\nu) J(\tau,\nu)}{|x(\tau,\nu) - \tilde{y}|} d\nu d\tau, \quad (3)$$

$$I_2 = \int_{\tau_0}^{\tau_1} \int_{\nu_0}^{\nu_1} \rho(\tau,\nu) L(\tau,\nu) J(\tau,\nu) \times \frac{\partial}{\partial \mathrm{n}_{\tilde{y}}} \frac{1}{|x(\tau,\nu) - \tilde{y}|} d\nu d\tau \quad (4)$$

$$I_3 = \int_{\tau_0}^{\tau_1} \int_{\nu_0}^{\nu_1} \rho(\tau,\nu) L(\tau,\nu) J(\tau,\nu) \times \frac{\partial}{\partial \mathrm{n}_{x}} \frac{1}{|x(\tau,\nu) - \tilde{y}|} d\nu d\tau \quad (5)$$

$$I_4 = \int_{\tau_0}^{\tau_1} \int_{\nu_0}^{\nu_1} \rho(\tau,\nu) L(\tau,\nu) J(\tau,\nu) \times \frac{\partial}{\partial \mathrm{n}_{\tilde{y}}} \frac{\partial}{\partial \mathrm{n}_{x}} \frac{1}{|x(\tau,\nu) - \tilde{y}|} d\nu d\tau \quad (6)$$

where $\rho(\tau, \iota)$ is the function of singularity of potential density, $L(\tau, \iota)$ is a basic function defined on P, $J(\tau, \iota)$ is the Jacobian of coordinate transformation. Construct the function of singularity of potential density

$$\rho(\tau, \iota) = \left(\frac{\tau_1 - \tau_0}{\tau - \tau_0}\right)^{\alpha_1} \left(\frac{\nu_1 - \nu_0}{\nu - \nu_0}\right)^{\alpha_2} \times \left(\frac{\tau_0 - \tau_1}{\tau - \tau_1}\right)^{\alpha_3} \left(\frac{\nu_0 - \nu_1}{\nu - \nu_1}\right)^{\alpha_4}.$$

Usually $mes\, P << mes\, S_i$, i.e. no more than two indexes α_j, $j = \overline{1,4}$, may be different from zero. If P is internal element of the grid $S_{h,i}$, $i = \overline{1,N}$, and point of observation

does not belong to P then function under the integral in (3)-(6) is smooth and for numerical integration the Gauss quadrature formula with constant weight various degrees of accuracy [21] is used.

3. Removal of the Singularity in the Density and Kernel of Simple Layer Potential

In order to simplify the presentation, we assume that $\Gamma_0 = [0,1]\times[0,1]$ is the unit square on the plane $x0y$. Parametric representation of such surface is given by

$$\begin{cases} x_1 = \tau, 0 \le \tau \le 1, \\ x_2 = \nu, 0 \le \nu \le 1, \\ x_3 = 0, \end{cases}$$

and Jacobian of coordinate transformation $J(\tau,\iota)=1$.

If P is the border element of the grid, e.g. $\tau_0 = \nu_0 = 0$, and point of observation does not belong to P then integral I_1 has the form

$$I_1 = \int_{\tau_0}^{\tau_1}\int_{\nu_0}^{\nu_1} \frac{\rho(\tau,\nu)L(\tau,\nu)J(\tau,\nu)}{|x(\tau,\nu)-\tilde{y}|}d\nu d\tau$$
$$= h_\tau^{\alpha_1} h_\nu^{\alpha_2} \int_0^{h_\tau} \tau^{-\alpha_1} \int_0^{h_\nu} \nu^{-\alpha_2} \frac{L(\tau,\nu)}{|x(\tau,\nu)-\tilde{y}|} d\nu\, d\tau.$$

In order to calculate I_1 sequentially the quadrature formula proposed in [22] is used:

$$\int_0^1 \varphi(\tau)\,\tau^{-\gamma} d\tau = 2\,(1-\gamma)^{-1}\,(2N+1)^{-1} \times \sum_{k=1}^{N} \phi(t_k) + R_N \quad (7)$$

where $t_k = \left(\frac{2k}{2N+1}\right)^{1/(1-\gamma)}$, $|R_N| \le \frac{1}{(2N+1)(1-\gamma)}$, $k = \overline{1,N}$. Choice of the formula (7) is explained by the simplicity of dependence on the singularity order γ.

Let P be the internal element of the grid and $\tilde{y} = (y_1, y_2, y_3) = (\tau_0, \nu_0, 0)$. In this case

$$I_1 = \int_{\tau_0}^{\tau_1}\int_{\nu_0}^{\nu_1} \frac{L(\tau,\nu)}{\sqrt{(\tau-\tau_0)^2+(\nu-\nu_0)^2}} d\nu d\tau\,. \quad (8)$$

Implement the change of variables in (8) by transition to polar coordinates system with center in the point (τ_0, ν_0)

$$\tau = \tau_0 + r\cos\theta,\ \nu = \nu_0 + r\sin\theta\,. \quad (9)$$

Then integral

$$I_1 = \int_0^{\pi/2}\int_0^{r_0(\theta)} L(\tau(r,\theta),\nu(r,\theta))\,dr d\theta$$

where $r_0(\theta)$ is function which determines the distance from the observation point to the opposite sides of the element, has no singularity. For its calculation the Gauss quadrature formula with constant weight is used.

If P is the border element of Γ_0, instead of (8) we have

$$I_1 = h_\tau^{\alpha_1} h_\nu^{\alpha_2} \int_{\tau_0}^{\tau_1}\int_{\nu_0}^{\nu_1} L(\tau,\nu)/[((\tau-\tau_0)^{\alpha_1} + (\nu-\nu_0)^{\alpha_2})\times$$
$$\times\sqrt{(\tau-\tau_0)^2+(\nu-\nu_0)^2}\,]\,d\nu d\tau$$

and in the case

$$L(\tau,\iota) = \frac{(\tau-\tau_0)(\nu-\nu_0)}{(\tau_1-\tau_0)(\nu_1-\nu_0)} \quad (10)$$

we obtain

$$I_1 = h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1} \int_{\tau_0}^{\tau_1}\int_{\nu_0}^{\nu_1} \frac{(\tau-\tau_0)^{\alpha_1-1}(\nu-\nu_0)^{\alpha_2-1}}{\sqrt{(\tau-\tau_0)^2+(\nu-\nu_0)^2}} d\nu d\tau\,.$$

Using the change of variables (9), we obtain

$$I_1 = \frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1}}{3-\alpha_1-\alpha_2} \int_0^{\pi/2} \cos^{\alpha_1-1}\theta \sin^{\alpha_2-1}\theta \times r_0^{3-\alpha_1-\alpha_2}(\theta)\,d\theta \quad (11)$$

Function under the integral in (11) has no singularity. To calculate it Gauss quadrature formula with constant weight is used. If

$$L(\tau,\iota) = \frac{(\tau-\tau_0)(\nu-\nu_1)}{(\tau_1-\tau_0)(\nu_0-\nu_1)} \quad (12)$$

we obtain

$$I_1 = \frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1}}{2-\alpha_1-\alpha_2} \int_0^{\pi/2} \frac{\cos^{\alpha_1-1}\theta}{\sin^{\alpha_2}\theta}\, r_0^{2-\alpha_1-\alpha_2}(\theta)\,d\theta$$
$$-\frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1}}{3-\alpha_1-\alpha_2} \int_0^{\pi/2} \cos^{\alpha_1-1}\theta \sin^{\alpha_2-1}\theta\, r_0^{3-\alpha_1-\alpha_2}(\theta)\,d\theta \quad (13)$$

The first integral in (13) is calculated by dividing the interval of integration into two intervals $[0,\varepsilon]$ and $[\varepsilon,\pi/2]$, $\varepsilon = \pi/180$, and replacing $\sin^{\alpha_2}\theta = \theta^{\alpha_2}$ on $[0,\varepsilon]$. Approximate value of the integral with the singularity is calculated using the quadrature formula (7), other integrals – by means of Gauss quadrature formula with constant weight.

Do the same for

$$L(\tau,\iota) = \frac{(\tau-\tau_1)(\nu-\nu_0)}{(\tau_0-\tau_1)(\nu_1-\nu_0)}\,. \quad (14)$$

In this case

$$I_1 = \frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1}}{2-\alpha_1-\alpha_2} \int_0^{\pi/2} \frac{\sin^{\alpha_2-1}\theta}{\cos^{\alpha_1}\theta} r_0^{2-\alpha_1-\alpha_2}(\theta) d\theta - \frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1}}{3-\alpha_1-\alpha_2} \int_0^{\pi/2} \cos^{\alpha_1-1}\theta \sin^{\alpha_2-1}\theta \times r_0^{3-\alpha_1-\alpha_2}(\theta) d\theta \quad (15)$$

The first integral in (15) is calculated by dividing the interval of integration into two intervals $[0, \pi/2-\varepsilon]$ and $[\pi/2-\varepsilon, \pi/2]$, $\varepsilon = \pi/180$, and replacing $\cos^{\alpha_1}\theta = = (\pi/2-\theta)^{\alpha_1}$ on $[\pi/2-\varepsilon, \pi/2]$. Approximate value of the integral with the singularity is calculated using the quadrature formula (7), other integrals – by means of Gauss quadrature formula with constant weight.

In case

$$L(\tau, \iota) = \frac{(\tau-\tau_1)(\nu-\nu_1)}{(\tau_0-\tau_1)(\nu_0-\nu_1)} \quad (16)$$

we obtain

$$I_1 = h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1} \int_0^{\pi/2} [\frac{r_0^{3-\alpha_1-\alpha_2}(\theta)}{3-\alpha_1-\alpha_2} \times \cos^{\alpha_1-1}\theta \sin^{\alpha_2-1}\theta - \frac{r_0^{2-\alpha_1-\alpha_2}(\theta)}{2-\alpha_1-\alpha_2} \times (\frac{h_\nu \cos^{\alpha_1-1}\theta}{\sin^{\alpha_2}\theta} + \frac{h_\tau \sin^{\alpha_2-1}\theta}{\cos^{\alpha_1}\theta}) + \frac{h_\tau h_\nu r_0^{1-\alpha_1-\alpha_2}(\theta)}{(1-\alpha_1-\alpha_2)\cos^{\alpha_1-1}\theta \sin^{\alpha_2-1}\theta}] d\theta \quad (17)$$

For the calculation of integrals with singularities in (17) the same techniques as for (13) and (15) are used. Similarly the removal of the singularity in the density of potential if the element of the grid is tangent to the edge of Γ_0 is performed.

4. Removal of the Singularity in the Density and Normal Derivative of the Kernel of Simple Layer Potential

Consider integral (4) if P is the internal element of the grid S_h and (τ_0, ν_0) is the observation point. In this case

$$I_2 = \int_{\tau_0}^{\tau_1} \int_{\nu_0}^{\nu_1} L(\tau, \nu)[(\tau-\tau_0)\cos\tilde{\alpha} + (\nu-\nu_0)\cos\tilde{\beta}] \times ((\tau-\tau_0)^2 + (\nu-\nu_0)^2)^{-3/2} d\nu d\tau) \quad (18)$$

where $\tilde{\alpha}$ is the angle between the vectors $\mathbf{e}_{\tilde{y}_1}$ and $\mathbf{n}_{\tilde{y}_1}$, $\tilde{\beta}$ is the angle between the vectors $\mathbf{e}_{\tilde{y}_2}$ and $\mathbf{n}_{\tilde{y}_2}$, the angle between the vectors $\mathbf{e}_{\tilde{y}_3}$ and $\mathbf{n}_{\tilde{y}_3}$ for the surface Γ_0 is equal to zero. Having changed variables (9) in integral (18) we obtain

$$I_2 = \int_0^{\pi/2} \int_0^{r_0(\theta)} \frac{L(\tau(r,\theta), \nu(r,\theta)\psi(\theta, \tilde{\alpha}, \tilde{\beta})}{r} dr d\theta$$

where $\psi(\theta, \tilde{\alpha}, \tilde{\beta}) = \cos\tilde{\alpha}\cos\theta + \cos\tilde{\beta}\sin\theta$. Then, in case (10) we have

$$I_2 = \frac{1}{2h_\tau h_\nu} \int_0^{\pi/2} \cos\theta \sin\theta \psi(\theta, \tilde{\alpha}, \tilde{\beta}) r_0^2(\theta) d\theta . \quad (19)$$

In case (12) we obtain

$$I_2 = \frac{1}{h_\tau h_\nu} \int_0^{\pi/2} \cos\theta \psi(\theta, \tilde{\alpha}, \tilde{\beta}) (h_\nu r_0(\theta) - \frac{1}{2} r_0^2(\theta) \sin\theta) d\theta \quad (20)$$

In case (34) we obtain

$$I_2 = \frac{1}{h_\tau h_\nu} \int_0^{\pi/2} \sin\theta \psi(\theta, \tilde{\alpha}, \tilde{\beta}) (h_\tau r_0(\theta) - \frac{1}{2} r_0^2(\theta) \cos\theta) d\theta \quad (21)$$

Functions under the integrals in (19)–(21) are smooth, and for calculation of integrals the Gauss quadrature formula with constant weight is used. Finally, in the case (16), we obtain

$$I_2 = \frac{1}{h_\tau h_\nu} \int_0^{\pi/2} \psi(\theta, \alpha, \beta) (\frac{1}{2} r_0^2(\theta) \cos\theta \sin\theta - r_0(\theta) (h_\nu \cos\theta + h_\tau \sin\theta) d\theta \int_0^{\pi/2} \psi(\theta, \alpha, \beta) \int_0^{r_0(\theta)} \frac{1}{r} dr d\theta \quad (22)$$

Take the function $q(\theta, \delta) = \ln((r_0(\theta) - \delta)/\delta)$ where parameter $\delta << r_0(\theta)$ as an approximate value of the integral $\int_0^{r_0(\theta)} \frac{1}{r} dr$.

If P is the boundary element of the grid S_h , e.g. $\tau_0 = \nu_0 = 0$, then instead of (19)–(21) we have integrals

$$I_2 = \frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1}}{2-\alpha_1-\alpha_2} \int_0^{\pi/2} r_0^{2-\alpha_1-\alpha_2}(\theta) \times \cos^{1-\alpha_1}\theta \sin^{1-\alpha_2}\theta \psi(\theta, \tilde{\alpha}, \tilde{\beta}) d\theta \quad (23)$$

$$I_2 = \frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2}}{1-\alpha_1-\alpha_2} \int_0^{\pi/2} \frac{\cos^{1-\alpha_1}\theta}{\sin^{\alpha_2}\theta} r_0^{1-\alpha_1-\alpha_2}(\theta) \times \psi(\theta,\tilde{\alpha},\tilde{\beta}) d\theta$$
$$- \frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1}}{2-\alpha_1-\alpha_2} \int_0^{\pi/2} r_0^{2-\alpha_1-\alpha_2}(\theta) \times \cos^{1-\alpha_1}\theta \sin^{1-\alpha_2}\theta \, \psi(\theta,\tilde{\alpha},\tilde{\beta}) d\theta \quad (24)$$

$$I_2 = \frac{h_\tau^{\alpha_1} h_\nu^{\alpha_2-1}}{1-\alpha_1-\alpha_2} \int_0^{\pi/2} \frac{\sin^{1-\alpha_2}\theta}{\cos^{\alpha_1}\theta} r_0^{1-\alpha_1-\alpha_2}(\theta) \times \psi(\theta,\tilde{\alpha},\tilde{\beta}) d\theta$$
$$- \frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1}}{2-\alpha_1-\alpha_2} \int_0^{\pi/2} r_0^{2-\alpha_1-\alpha_2}(\theta) \times \cos^{1-\alpha_1}\theta \sin^{1-\alpha_2}\theta \, \psi(\theta,\tilde{\alpha},\tilde{\beta}) d\theta \quad (25)$$

Integrals with singularities in (24) and (25) are calculated similarly to (13) and (15). Instead of (23) we have

$$I_2 = \frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1}}{2-\alpha_1-\alpha_2} \int_0^{\pi/2} r_0^{2-\alpha_1-\alpha_2}(\theta) \cos^{1-\alpha_1}\theta \times \sin^{1-\alpha_2}\theta \, \psi(\theta,\tilde{\alpha},\tilde{\beta}) d\theta$$
$$- \frac{h_\tau^{\alpha_1-1} h_\nu^{\alpha_2-1}}{1-\alpha_1-\alpha_2} \times \int_0^{\pi/2} r_0^{1-\alpha_1-\alpha_2}(\theta) \left(\frac{h_\tau \sin^{1-\alpha_2}\theta}{\cos^{\alpha_1}\theta} + \frac{h_\nu \cos^{1-\alpha_1}\theta}{\sin^{\alpha_2}\theta}\right) \times \psi(\theta,\tilde{\alpha},\tilde{\beta}) d\theta \quad (26)$$
$$+ h_\tau^{\alpha_1} h_\nu^{\alpha_2} \int_0^{\pi/2} \cos^{\alpha_1}\theta \times \sin^{\alpha_2}\theta \int_0^{r_0(\theta)} \frac{dr}{r^{1+\alpha_1+\alpha_2}} d\theta$$

Take the function

$$\tilde{q}(\theta,\tilde{\delta}) = (r_0^{\alpha_1+\alpha_2}(\theta) - \tilde{\delta}^{\alpha_1+\alpha_2}) / (\alpha_1+\alpha_2) \tilde{\delta}^{\alpha_1+\alpha_2} r_0^{\alpha_1+\alpha_2}(\theta)$$

where parameter $\tilde{\delta} << r_0(\theta)$ as an approximate value of the integral $\int_0^{r_0(\theta)} \frac{dr}{r^{1+\alpha_1+\alpha_2}}$.

The removal of the singularity in the density of potential if the element of the grid is tangent to the edge of Γ_0 is performed similarly.

5. Removal of the Singularity in the Density and Kernel of Double Layer Potential

Consider the integral (5). In this case

$$I_3 = \int_{\tau_0}^{\tau_1} \int_{\nu_0}^{\nu_1} L(\tau,\nu)[(\tau-\tau_0)\cos\alpha + (\nu-\nu_0)\cos\beta] \times ((\tau-\tau_0)^2 + (\nu-\nu_0)^2)^{-3/2} d\nu d\tau \quad (27)$$

where α is the angle between the vectors $\mathbf{e}_{x_1}$ and $\mathbf{n}_{x_1}$, β is the angle between the vectors $\mathbf{e}_{x_2}$ and $\mathbf{n}_{x_2}$, the angle between the vectors $\mathbf{e}_{x_3}$ and $\mathbf{n}_{x_3}$ for the surface Γ_0 is equal zero. Having changed variables (9) in integral (27), we obtain

$$I_3 = \int_0^{\pi/2} \int_0^{r_0(\theta)} \frac{L(\tau(r,\theta),\nu(r,\theta)\, \phi(\theta,\alpha,\beta)}{r} dr d\theta \quad (28)$$

where $\phi(\theta,\alpha,\beta) = \cos\alpha\cos\theta + \cos\beta\sin\theta$. In order to calculate the integral (28) the same methods as in previous subsection are used.

Similarly, the integral I_4 reduces to the form

$$I_4 = \int_0^{\pi/2} \int_0^{r_0(\theta)} L(\tau(r,\theta),\nu(r,\theta) \times \chi(\theta,\alpha,\beta,\tilde{\alpha},\tilde{\beta}) / r^2 \, dr d\theta \quad (29)$$

where

$$\chi(\theta,\alpha,\beta,\tilde{\alpha},\tilde{\beta}) = (2\cos^2\theta - \sin^2\theta)\cos\alpha\cos\tilde{\alpha}$$
$$+ 3\cos\theta\sin\theta\cos\alpha\cos\tilde{\beta} + 3\cos\theta\sin\theta\cos\beta\cos\tilde{\alpha}$$
$$+ (2\sin^2\theta - \cos^2\theta)\cos\beta\cos\tilde{\beta}$$

In order to calculate the integral (29) the quadrature formulas for singular integrals proposed in [23] is used. We can also use the Galerkin method for numerical solution of corresponding integral equations, although it involves significant increase in the computational expenditures.

Let Γ_0 be the part of the boundary surface Γ with parametric representation

$$\begin{cases} x_1 = x_1(\tau,\nu), \\ x_2 = x_2(\tau,\nu), \quad 0 \le \tau \le a, 0 \le \nu \le b, \\ x_3 = x_3(\tau,\nu), \end{cases}$$

and rectangular grid S_h be generated in domain of parameters $S = [0,a] \times [0,b] \in R^2$. Denote $R(\tau,\nu) = 1/|x(\tau,\nu) - \tilde{y}|$, $\tau,\nu \in P$, $\tilde{y}$ is a fixed point of element of the grid P. In case $\tilde{y} \to x(\tau,\nu)$ the integral (3) has singularity. To remove the latter, make the change of

variables (9) where τ_0, ν_0 are parametric coordinates of the point $\tilde{y}$. Then $R(\tau,\nu)d\tau d\nu = = rR(\tau(r,\theta),\nu(r,\theta))drd\theta \to K(\theta)drd\theta$ when $r \to 0$ and the integral (3) can be singular only due to potential density. The latter is easy removed when the density is approximated by the piecewise-linear elements and the quadrature formula (7) is used to calculate (3).

In general, the reduction of integral (3) to a single integral with a replacement (9), applied for the flat part of the boundary surface is impossible. Therefore, to calculate the integral

$$I = \int_0^{\pi/2} \int_0^{r_0(\theta)} f(r,\theta) r^{-\alpha} drd\theta$$

consistently the Gauss quadrature formula and one of the previously considered methods for the calculation of singular integrals which depends on the value of α are used. If step of the grid on the surface Γ_0 is small enough then element $P \in S_h$ can be replaced by a flat. In this case we can use algorithms considered above for removal of singularities for the flat surface. Proposed approach can be easily extended to the case of nonrectangular grids, higher-order finite element approximations, and other projection methods for numerical solution of the system of integral equations (1).

6. Results and Discussion

Consider several examples with known analytical solution that are used to study numerical solution distortions that occur close to edges and corner points of boundary surface. They allow to examine the effectiveness of application of methods proposed for singularities removal in kernels and densities of potentials.

Let $G = [0,1] \times [0,1] \times [0,1]$ be the unit cube and $\Gamma = = \partial G \cup \Gamma_0$ where Γ_0 is a flat tired surface $[0.25, 0.75] \times [0.25, 0.75]$ which lies in the plane $y_3 = 0.5$. Assume that the unknown function is continuous when crossing ∂G and Γ_0. In order to illustrate the effectiveness of the proposed algorithms the following boundary value problems were solved.

Problem 1. Find function $u \in G$, $\Delta u = 0$ in G which satisfied boundary condition $u|_\Gamma = 1$. Solution of this problem is known, namely $u = 1$ in G. In all parts of Γ (the sides of the cube and the surface Γ_0), the rectangular grid with $h_\tau = h_\nu$ =0.125 was generated. The solution was proposed in the form of simple layer potential with a piecewise linear approximation of the smooth part of the unknown potential density. The values α_i, $i = \overline{1,4}$, are equal to 0.5. For the solution of integral equation the collocation method is used with the observation points coinciding with the grid nodes. The dimension of the system of collocation equations N_1==458. To calculate integrals (3) first only Gauss quadrature formula of the fourth order with constant weight without removal of singularities was used. The accuracy of the approximate solution in G when approaching the middle of the edge of the cube was equal 2.1%, when approaching the middle of the edge Γ_0 – 3.5%, when approaching the corner points of ∂G and Γ_0 – 5.8%. When Gauss quadrature formula of the fourth order with constant coefficients as well as proposed algorithms for removing singularities were used, the accuracy of approximate solution in G while approaching the middle of the cube edge was equal to 0.3 %, when approaching the middle of the edge Γ_0 – 0.5 %, when approaching the corner points of boundary surface – 0.9 %. Result of similar accuracy may be achieved without usage of proposed algorithms for removal singularities. For this, the grid with step $h_\tau = h_\nu$ =0.0625 has to generated on the boundary surface. The dimension of the system of collocation equations in this case N_2=1714. The number of operations required for the formation of the system of collocation equations in second case is over 14 times higher than number of operations required to generate appropriate system in first case. The number of operations required for solution of the system of collocation equations with the use of Gauss method $O(N_2^3)$ is over 52 times higher than number of operations $O(N_1^3)$ required for solution of the system of linear algebraic equations in case of singularities removal.

Problem 2 differs from problem 1 only in boundary value condition on the surface Γ_0 which is assumed to be $\left.\frac{\partial u}{\partial \mathbf{n}}\right|_{\Gamma_0} = 0$. To calculate integrals (3) and (4), first only Gauss quadrature formula of the fourth-order with constant weight without singularities removal was used. The accuracy of the approximate solution in G when approaching the middle of the edge of the cube was equal 3.1%, when approaching the middle of the edge Γ_0 – 5.7%, when approaching the corner points of ∂G and Γ_0 – 8.2%.

When Gauss quadrature formula of the fourth order with constant coefficients as well as proposed algorithms for removing singularities were used for integrals (3) and (4), the accuracy of approximate solution in G while approaching the middle of the cube edge was equal 0.6 %, when approaching the middle of the edge Γ_0 – 0.9%, when approaching the corner points of ∂G and Γ_0 – 1.2%.

7. Conclusions

Boundary integral equation method is a powerful tool for solving many problems of mathematical physics. The accuracy of the numerical solution of these equations depends essentially on the presence of singularities in the kernel and density of potential. The algorithms are developed for removal singularities in densities and kernels of simple and double layer potentials and its normal derivatives near the edges and corner points of the boundary surface using the properties of finite element approximation. Numerical

experiments demonstrate the high efficiency of approach proposed. These methods can be used to improve the accuracy of the numerical solution of integral and integral-differential equations with singularities equivalent to many other problems of mathematical physics.

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