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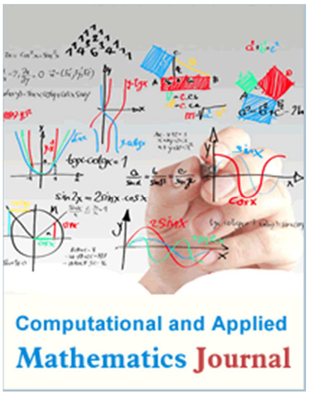

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Transmuted Gompertz Distribution

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Abstract

In this paper, we introduce a new distribution called transmuted Gompertz distribution (TGD). Some properties of this distribution will be discussed. The r^{th} traditional moments, TL-moments, L-moments are derived. The maximum likelihood estimation of the unknown parameters is discussed. Real data are used to determine whether the TGD is better than Gompertz distribution (GD) or not.

1. Introduction

Transmuted distributions can be obtained by adding a real number λ ($|\lambda|\le 1$) to the cumulative distribution function (*CDF*). i.e, if we have *CDF* $G(x)$ of any random variable X, then the function

$$F(x)=(1+\lambda)G(x)-\lambda\left[G(x)\right]^2,\ |\lambda|\le 1, \tag{1}$$

is called a transmuted distribution. See Ref.[1].

The probability density function (*PDF*) is given by

$$f(x)=\frac{dF(x)}{dx}=\left[(1+\lambda)-2\lambda G(x)\right]g(x), \tag{2}$$

where $g(x)$ is the *PDF* of base distribution.

A random variable X is said to have a Gompertz distribution with two parameters α and θ if its *PDF* is in the form

$$g(x)=\alpha\theta e^{\alpha x-\theta\left(e^{\alpha x}-1\right)};\ x\ge 0,\ \alpha \text{ and } \theta>0. \tag{3}$$

Then the *CDF* of $g(x)$ in (3) is given by

$$G(x)=1-e^{-\theta\left(e^{\alpha x}-1\right)};\ x\ge 0,\ \alpha \text{ and } \theta>0. \tag{4}$$

Many transmuted distributions are proposed. Ref. [2] presented a new generalization of Weibull distribution called the transmuted Weibull distribution. Ref. [3] proposed and studied the various structural properties of the transmuted Rayleigh distribution. Ref.[4] introduced the transmuted modified Weibull distribution. Transmuted Lomax distribution is presented by Ref.[5]. Ref.[6] introduce transmuted Pareto distribution. Transmuted Generalized Linear Exponential Distribution introduced by Ref.[7] among other. In this paper we present a new generalization of the Gompertz distribution called the transmuted

Gompertz distribution. The rest of the paper is organized as follows. In Section 2 we introduce transmuted Gompertz distribution (TGD). The traditional moments for TGD are given in Section 3. In Section 4 we discussed the parameters estimators using maximum likelihood method. In Section 5 we demonstrate the order statistics for TGD. The r^{th} TL-moments with different trimmed are given in Section 6. The r^{th} L-moments was introduced In Section 7.

Finally, we use a real data set to show that the TGD can be a better model than one based on the GD in Section 8.

2. Transmuted Gompertz Distribution

Using (1) and (4), we can define the CDF of transmuted Gompertz distribution (TGD) as follows

$$F(x)=(1+\lambda)\left[1-e^{-\theta\left(e^{\alpha x}-1\right)}\right]-\lambda\left[1-e^{-\theta\left(e^{\alpha x}-1\right)}\right]^2$$
$$=\left[1-e^{-\theta\left(e^{\alpha x}-1\right)}\right]\left[1+\lambda e^{-\theta\left(e^{\alpha x}-1\right)}\right];$$
$$x\ge 0,\ |\lambda|\le 1, \alpha \text{ and } \theta>0\,. \tag{5}$$

The *PDF* of *TGD* is

$$f(x)=\alpha\theta e^{\alpha x}e^{-\theta\left(e^{\alpha x}-1\right)}\left[1-\lambda+2\lambda e^{-\theta\left(e^{\alpha x}-1\right)}\right];\quad x\ge 0,\ |\lambda|\le 1, \alpha \text{ and } \theta>0 \tag{6}$$

Figure 1 and Figure 2 depict the behavior of the distribution for some parameter values.

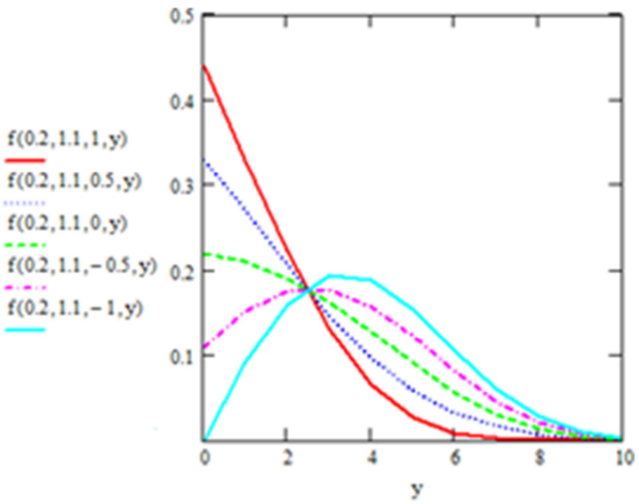

Figure 1. *plots of PDF,* $f(\alpha,\theta,\lambda,y)$.

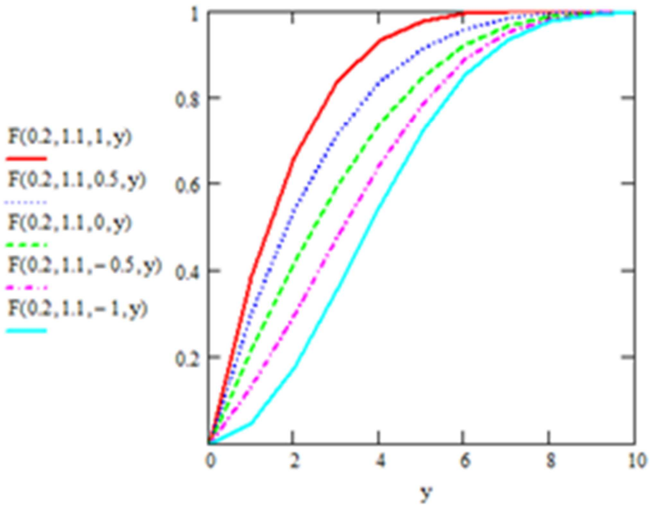

Figure 2. *plots of CDF* $F(\alpha,\theta,\lambda,y)$.

The reliability function $R(x)$, the hazard rate function (*HRF*) $h_\lambda(x)$ and the reversed hazard rate function (*RHRF*) $h_\lambda^*(x)$ for TGD are in the following forms:

$$R(x)=e^{-\theta\left(e^{\alpha x}-1\right)}\left[1-\lambda+\lambda e^{-\theta\left(e^{\alpha x}-1\right)}\right];\quad x\ge 0,\ |\lambda|\le 1, \alpha \text{ and } \theta>0, \tag{7}$$

$$h_\lambda(x)=\frac{h(x)\left[1-\lambda+2\lambda e^{-\theta\left(e^{\alpha x}-1\right)}\right]}{\left[1-\lambda+\lambda e^{-\theta\left(e^{\alpha x}-1\right)}\right]};\quad x\ge 0,\ |\lambda|\le 1, \alpha \text{ and } \theta>0\,, \tag{8}$$

and

$$h_\lambda^*(x)=h^*(x)-\frac{h^*(x)\lambda e^{-\theta\left(e^{\alpha x}-1\right)}}{\left[1-\lambda+2\lambda e^{-\theta\left(e^{\alpha x}-1\right)}\right]};\quad x\ge 0,\ |\lambda|\le 1, \alpha \text{ and } \theta>0, \tag{9}$$

where $h(x)$ and $h^*(x)$ are the *HRF* and *RHRF* for the Gompertz distribution respectively. Figure 3 depict the behavior of *HRF* for some parameter values.

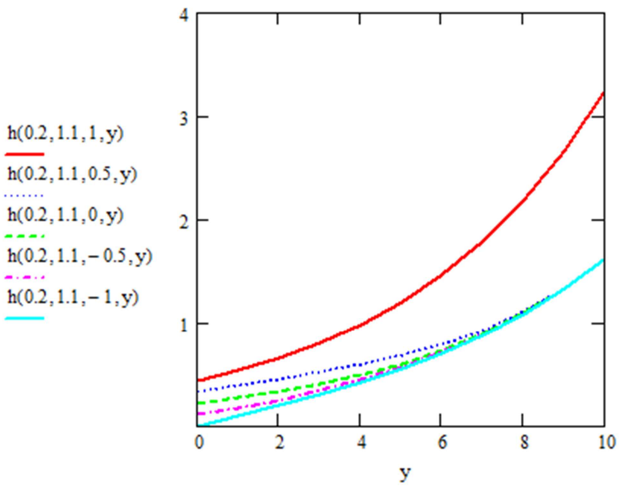

***Figure 3.** plot of HRF* $h(\alpha,\theta,\lambda,y)$.

Figure 3. show that $h_\lambda(y)$ is monotonically increasing with y.

The median of TGD is

$$x_{0.5}=\frac{1}{\alpha}\ln\left\{-\frac{1}{\theta}\ln\left[\frac{\lambda-1+\sqrt{1+\lambda^2}}{2\lambda}\right]+1\right\}.$$

3. Traditional Moments for TGD

The moments generating function for TGD is

$$\begin{aligned}M(t)&=E\left(e^{tX}\right)\\&=\alpha\theta\int_0^\infty e^{tx}e^{\alpha x}e^{-\theta\left(e^{\alpha x}-1\right)}\left[1-\lambda+2\lambda e^{-\theta\left(e^{\alpha x}-1\right)}\right]dx\end{aligned}$$

Let $y=\theta\left(e^{\alpha x}-1\right)\Rightarrow dy=\alpha\theta e^{\alpha x}dx$

$$x=\frac{1}{\alpha}\left[\ln(y+\theta)-\ln(\theta)\right] \text{ and } 0\le y<\infty.$$

$$M(t)=\int_0^\infty\left[\frac{y+\theta}{\theta}\right]^{\frac{t}{\alpha}}e^{-y}\left[1-\lambda+2\lambda e^{-y}\right]dy$$

Let $z=\frac{y+\theta}{\theta}\Rightarrow dy=\theta dz$, $y=\theta z-\theta$ and $1\le z<\infty$.

$$\begin{aligned}M(t)=\theta^{-\frac{t}{\alpha}}(1-\lambda)e^{\theta}&\left[\Gamma\left(\frac{t}{\alpha}+1\right)-\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{\frac{t}{\alpha}+1+i}}{i!\left(\frac{t}{\alpha}+1+i\right)}\right]+\\(2\theta)^{-\frac{t}{\alpha}}\lambda e^{2\theta}&\left[\Gamma\left(\frac{t}{\alpha}+1\right)-\sum_{i=0}^{\infty}\frac{(-1)^i(2\theta)^{\frac{t}{\alpha}+1+i}}{i!\left(\frac{t}{\alpha}+1+i\right)}\right]\end{aligned}\tag{10}$$

The first two moments about zero are

$$\begin{aligned}\mu_1'&=M'(0)\\&=\frac{(1-\lambda)e^{\theta}}{\alpha}\left[-\ln(\theta)-\gamma+\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{i+1}}{(i+1)!(i+1)}\right]\\&+\frac{\lambda e^{2\theta}}{\alpha}\left[-\ln(2\theta)-\gamma+\sum_{i=0}^{\infty}\frac{(-1)^i(2\theta)^{i+1}}{(i+1)!(i+1)}\right]\\\mu_1'&=-\frac{(1-\lambda)e^{\theta}}{\alpha}Ei(-\theta)-\frac{\lambda e^{2\theta}}{\alpha}Ei(-2\theta),\end{aligned}\tag{11}$$

where γ is Euler constant and $Ei(-z)=-\int_z^\infty t^{-1}e^{-t}dt$.

$$\begin{aligned}\mu_2'&=M''(0)\\&=\frac{(1-\lambda)e^{\theta}}{\alpha^2}\left[\left[\ln(\theta)\right]^2+2\gamma\ln(\theta)+\frac{\pi^2}{6}+\gamma^2\right.\\&\left.-\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{i+1}}{(i+1)!(i+1)^2}\right]+\frac{\lambda e^{2\theta}}{\alpha^2}\left[\left[\ln(2\theta)\right]^2\right.\\&\left.+2\gamma\ln(2\theta)+\frac{\pi^2}{6}+\gamma^2-\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{i+1}}{(i+1)!(i+1)^2}\right]\end{aligned}$$

The variance is

$$\begin{aligned}Var(x)=\frac{(1-\lambda)e^{\theta}}{\alpha^2}&\left[\left[\ln(\theta)\right]^2+2\gamma\ln(\theta)+\frac{\pi^2}{6}\right.\\&\left.+\gamma^2-2\sum_{i=0}^{\infty}\frac{(-1)^i\theta^{i+1}}{(i+1)!(i+1)^2}\right]+\frac{\lambda e^{2\theta}}{\alpha^2}\\&\left[\left[\ln(2\theta)\right]^2+2\gamma\ln(2\theta)+\frac{\pi^2}{6}+\gamma^2\right.\\&\left.-2\sum_{i=0}^{\infty}\frac{(-1)^i(2\theta)^{i+1}}{(i+1)!(i+1)^2}\right]\\&-\left[\frac{(1-\lambda)e^{\theta}}{\alpha}Ei(-\theta)+\frac{\lambda e^{2\theta}}{\alpha}Ei(-2\theta)\right]^2\end{aligned}\tag{12}$$

4. Parameter Estimators

In this section, we consider maximum likelihood estimators (MLE) of TGD. Let $x_1,x_2,\ldots,x_n$ be a random sample of size n from TGD, then the log-likelihood function $L(\alpha,\theta,\lambda)$ can be written as

$$\begin{aligned}\ln L(\alpha,\theta,\lambda)\propto n\left[\ln(\theta)+\ln(\alpha)\right]-\theta\sum_{i=1}^{n}\left(e^{\alpha x_i}-1\right)\\+\alpha\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\ln\left[1-\lambda+2\lambda e^{-\theta\left(e^{\alpha x_i}-1\right)}\right]\end{aligned}\tag{13}$$

The normal equations become

$$\frac{\partial \ln L}{\partial \alpha}=\frac{n}{\alpha}-\theta\sum_{i=1}^{n}x_i e^{\alpha x_i}+\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{2\lambda\theta x_i e^{\alpha x_i}e^{-\theta\left(e^{\alpha x_i}-1\right)}}{1-\lambda+2\lambda e^{-\theta\left(e^{\alpha x_i}-1\right)}}, \tag{14}$$

$$\frac{\partial \ln L}{\partial \theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\left(e^{\alpha x_i}-1\right)-\sum_{i=1}^{n}\frac{2\lambda\left(e^{\alpha x_i}-1\right)e^{-\theta\left(e^{\alpha x_i}-1\right)}}{1-\lambda+2\lambda e^{-\theta\left(e^{\alpha x_i}-1\right)}}, \tag{15}$$

$$\frac{\partial \ln L}{\partial \lambda}=\sum_{i=1}^{n}\frac{2e^{-\theta\left(e^{\alpha x_i}-1\right)}-1}{1-\lambda+2\lambda e^{-\theta\left(e^{\alpha x_i}-1\right)}}. \tag{16}$$

The MLE of α, θ and λ can be obtain by solving the equations (14), (15), and (16) using $\frac{\partial \ln L}{\partial \alpha}=0$, $\frac{\partial \ln L}{\partial \theta}=0$ and $\frac{\partial \ln L}{\partial \lambda}=0$.

5. Order Statistics for TGD

In this section, we drive the single order statistics for TGD. The probability density function of $X_{a:n}$ is:

$$f_{a:n}(x)=\frac{n!}{(a-1)!(n-a)!}\left[F(x)\right]^{a-1}f(x)\left[1-F(x)\right]^{n-a}$$

Special cases:

When $n=2$, we get *PDF* of two *iid* random variables X_1, X_2 as follows:

$$f_{a:2}(x)=\frac{2}{(a-1)!(2-a)!}\left[F(x)\right]^{a-1}f(x)\left[1-F(x)\right]^{2-a};a=1,2$$

The *PDF* of $\min(X_1,X_2)$ is ($a=1$)

$$f_{1:2}(x)=2f(x)\left[1-F(x)\right]. \tag{17}$$

and the *PDF* of $\max(X_1,X_2)$ ($a=2$) is

$$f_{2:2}(x)=2F(x)f(x). \tag{18}$$

(17) and (18) can be obtained from (2), which $\lambda=1$ and $\lambda=-1$ respectively. This result means that if X_1, X_2 are two *iid* random variables have *DF* $F(x)$. Then the DF of $\min(X_1,X_2)$ and $\max(X_1,X_2)$ are the corresponding transmuted distribution of $F(x)$ at $\lambda=1$ and $\lambda=-1$ respectively.

The first moment of order statistics are in the following form

$$\mu'_{a:n}=E\left(X_{a:n}\right)=\frac{n!}{(a-1)!(n-a)!}\int_{-\infty}^{\infty}x\left[F(x)\right]^{a-1}f(x)\left[1-F(x)\right]^{n-a}dx$$

To find the first moment of order statistics for TGD, we use $F(x)$ and $f(x)$ in (5) and (6) as:

$$\mu'_{a:n}=\frac{n!}{(a-1)!(n-a)!}\alpha\theta\int_{0}^{\infty}x\left[1-e^{-\theta\left(e^{\alpha x}-1\right)}\right]^{a-1}\left[1+\lambda e^{-\theta\left(e^{\alpha x}-1\right)}\right]^{a-1}e^{\alpha x}e^{-\theta(n-a+1)\left(e^{\alpha x}-1\right)}\left[1-\lambda+2\lambda e^{-\theta\left(e^{\alpha x}-1\right)}\right]\left[1-\lambda+\lambda e^{-\theta\left(e^{\alpha x}-1\right)}\right]^{n-a}dx$$

Let $y=\theta\left(e^{\alpha x}-1\right)\Rightarrow dy=\alpha\theta e^{\alpha x}dx$, $x=\frac{1}{\alpha}\left[\ln(y+\theta)-\ln(\theta)\right]$ and $0\le y<\infty$.

$$\begin{aligned}\mu'_{a:n}&=\frac{n!}{\alpha(a-1)!(n-a)!}\int_{0}^{\infty}\left[\ln(y+\theta)-\ln(\theta)\right]\left(1-e^{-y}\right)^{a-1}\left(1+\lambda e^{-y}\right)^{a-1}e^{-y(n-a+1)}\left(1-\lambda+2\lambda e^{-y}\right)\left(1-\lambda+\lambda e^{-y}\right)^{n-a}dy\\&=\frac{n!}{\alpha(a-1)!(n-a)!}\sum_{i=0}^{a-1}\sum_{j=0}^{a-1}\sum_{l=0}^{n-a}\binom{a-1}{i}\binom{a-1}{j}\binom{n-a}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{n-a-l}\\&\quad\int_{0}^{\infty}\left[\ln(y+\theta)-\ln(\theta)\right]e^{-(n-a+i+j+l+1)y}\left(1-\lambda+2\lambda e^{-y}\right)dy\\&=\frac{n!}{\alpha(a-1)!(n-a)!}\sum_{i=0}^{a-1}\sum_{j=0}^{a-1}\sum_{l=0}^{n-a}\binom{a-1}{i}\binom{a-1}{j}\binom{n-a}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{n-a-l}\left\{\frac{(\lambda-1)e^{\theta(n-a+i+j+l+1)}}{n-a+i+j+l+1}\right.\\&\quad\left.Ei\left[-\theta(n-a+i+j+l+1)\right]-\frac{2\lambda}{n-a+i+j+l+2}e^{\theta(n-a+i+j+l+2)}Ei\left[-\theta(n-a+i+j+l+2)\right]\right\},\end{aligned} \tag{19}$$

where $Ei(-z) = -\int_{z}^{\infty} t^{-1} e^{-t} dt$.

6. Population TL-moments for TGD

In this section, we investigate the population TL-moment of order r for TGD. TL-moments are more robust than traditional moments and exist even if the distribution does not have a mean, for example the TL-moments are existed for Cauchy distribution. The r^{th} TL-moments is given in the following formula Ref.[10]

$$L_r^{(s,t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left(Y_{r+s-k:r+s+t}\right), \qquad (20)$$

Where r, s and t takes the values 1, 2, 3, …. We have note that the r^{th} L-moments can be obtained by taking $t = s = 0$. Ref.[11] are introduced an important relation between the first TL-moments $L_1^{(s,t)}$ and the r^{th} TL-moments $L_r^{(s,t)}$ as follows:

$$L_r^{(s,t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} L_1^{(r+s-k-1,t+k)}, \quad r = 2,3,\ldots \qquad (21)$$

Using formula (20) and the function (19), the TL-moment of order r for TGD will be as follows

$$L_r^{(s,t)} = \frac{1}{r}\sum_{k=0}^{r-1}\sum_{i=0}^{(r+s-k-1)}\sum_{j=0}^{(r+s-k-1)}\sum_{l=0}^{t+k} \frac{(-1)^{k+i}\lambda^{j+l}}{\alpha(t+k)!}\frac{(1-\lambda)^{s+t-l}(r+s+t)!}{(r+s-k-1)!}\binom{r+s-k-1}{i}\binom{r-1}{k}\binom{r+s-k-1}{j}\binom{t+k}{l}$$
$$\left\{\frac{(\lambda-1)e^{\theta(s+t+i+j+l+1)}}{s+t+i+j+l+1}Ei\left[-\theta(s+t+i+j+l+1)\right] - \frac{2\lambda e^{\theta(s+t+i+j+l+2)}}{s+t+i+j+l+2}Ei\left[-\theta(s+t+i+j+l+2)\right]\right\}, \qquad (22)$$

Where $r, s, t = 1, 2, 3, \ldots$ and $s + t < n$.
The first three TL-moments can be obtained by taking $r = 1$ in (22) and using (21) as follows

$$L_1^{(s,t)} = \frac{(s+t+1)!}{\alpha s!t!}\sum_{i=0}^{s}\sum_{j=0}^{s}\sum_{l=0}^{t}\binom{s}{i}\binom{s}{j}\binom{t}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{s+t-l}$$
$$\left\{\frac{(\lambda-1)e^{\theta(s+t+i+j+l+1)}}{s+t+i+j+l+1}Ei\left[-\theta(s+t+i+j+l+1)\right] - \frac{2\lambda e^{\theta(s+t+i+j+l+2)}}{s+t+i+j+l+2}Ei\left[-\theta(s+t+i+j+l+2)\right]\right\}, \qquad (23)$$

$$L_2^{(s,t)} = \frac{1}{2}\left[L_1^{(s+1,t)} - L_1^{(s,t+1)}\right] = \frac{(s+t+2)!}{2\alpha s!t!}\left\{\frac{1}{s+1}\sum_{i=0}^{s+1}\sum_{j=0}^{s+1}\sum_{l=0}^{t}\binom{s+1}{i}\binom{s+1}{j}\binom{t}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{s+t-l+1}\right.$$
$$\left. - \frac{1}{t+1}\sum_{i=0}^{s}\sum_{j=0}^{s}\sum_{l=0}^{t+1}\binom{s}{i}\binom{s}{j}\binom{t+1}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{s+t-l+1}\right\}$$
$$\left\{\frac{(\lambda-1)e^{\theta(s+t+i+j+l+2)}}{s+t+i+j+l+2}Ei\left[-\theta(s+t+i+j+l+2)\right] - \frac{2\lambda e^{\theta(s+t+i+j+l+3)}}{s+t+i+j+l+3}Ei\left[-\theta(s+t+i+j+l+3)\right]\right\}, \qquad (24)$$

and

$$L_3^{(s,t)} = \frac{1}{3}\left[L_1^{(s+2,t)} - 2L_1^{(s+1,t+1)} + L_1^{(s,t+2)}\right]$$
$$= \frac{(s+t+3)!}{3\alpha s!t!}\left\{\frac{1}{(s+2)(s+1)}\sum_{i=0}^{s+2}\sum_{j=0}^{s+2}\sum_{l=0}^{t}\binom{s+2}{i}\binom{s+2}{j}\binom{t}{l}(-1)^i(1-\lambda)^{s+t-l+2}\lambda^{j+l}\right.$$
$$-\frac{2}{(s+1)(t+1)}\sum_{i=0}^{s+1}\sum_{j=0}^{s+1}\sum_{l=0}^{t+1}\binom{s+1}{i}\binom{s+1}{j}\binom{t+1}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{s+t-l+2}$$
$$\left.+\frac{1}{(t+2)(t+1)}\sum_{i=0}^{s}\sum_{j=0}^{s}\sum_{l=0}^{t+2}\binom{s}{i}\binom{s}{j}\binom{t+2}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{s+t-l+2}\right\}$$
$$\left\{\frac{(\lambda-1)e^{\theta(s+t+i+j+l+3)}}{s+t+i+j+l+3}Ei\left[-\theta(s+t+i+j+l+3)\right] - \frac{2\lambda e^{\theta(s+t+i+j+l+4)}}{s+t+i+j+l+4}Ei\left[-\theta(s+t+i+j+l+4)\right]\right\}, \qquad (25)$$

Special cases:

Symmetric trimmed $(s=t)$:

In this case the r^{th} and the first three TL-moments are in the forms:

$$L_r^{(s,s)}=\frac{1}{r}\sum_{k=0}^{r-1}\sum_{i=0}^{(r+s-k-1)}\sum_{j=0}^{(r+s-k-1)}\sum_{l=0}^{s+k}\frac{(-1)^{k+i}(1-\lambda)^{2s-l}}{\alpha(r+s-k-1)!}\times\frac{\lambda^{j+l}(r+2s)!}{(s+k)!}\binom{r-1}{k}\binom{r+s-k-1}{i}\times\binom{r+s-k-1}{j}\binom{s+k}{l}\left\{\frac{(\lambda-1)e^{\theta(2s+i+j+l+1)}}{2s+i+j+l+1}\times Ei\left[-\theta(2s+i+j+l+1)\right]-\frac{2\lambda e^{\theta(2s+i+j+l+2)}}{2s+i+j+l+2}\times Ei\left[-\theta(2s+i+j+l+2)\right]\right\}, \tag{26}$$

$$L_1^{(s,s)}=\frac{(2s+1)!}{\alpha(s!)^2}\sum_{i=0}^{s}\sum_{j=0}^{s}\sum_{l=0}^{s}\binom{s}{i}\binom{s}{j}\binom{s}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{2s-l}\left\{\frac{(\lambda-1)e^{\theta(2s+i+j+l+1)}}{2s+i+j+l+1}Ei\left[-\theta(2s+i+j+l+1)\right]-\frac{2\lambda e^{\theta(2s+i+j+l+2)}}{2s+i+j+l+2}\times Ei\left[-\theta(2s+i+j+l+2)\right]\right\}, \tag{27}$$

$$L_2^{(s,s)}=\frac{1}{2}\left[L_1^{(s+1,s)}-L_1^{(s,s+1)}\right]=\frac{(2s+2)!}{2\alpha(s!)^2}\left\{\frac{1}{s+1}\sum_{i=0}^{s+1}\sum_{j=0}^{s+1}\sum_{l=0}^{s}\binom{s+1}{i}\binom{s+1}{j}\binom{s}{l}(-1)^i\lambda^{j+l}\times(1-\lambda)^{2s-l+1}-\frac{1}{s+1}\sum_{i=0}^{s}\sum_{j=0}^{s}\sum_{l=0}^{s+1}\binom{s}{i}\times\binom{s}{j}\binom{s+1}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{2s-l+1}\right\}\left\{\frac{(\lambda-1)e^{\theta(2s+i+j+l+2)}}{2s+i+j+l+2}Ei\left[-\theta(2s+i+j+l+2)\right]-\frac{2\lambda e^{\theta(2s+i+j+l+3)}}{2s+i+j+l+3}Ei\left[-\theta(2s+i+j+l+3)\right]\right\}, \tag{28}$$

and

$$L_3^{(s,s)}=\frac{1}{3}\left[L_1^{(s+2,s)}-2L_1^{(s+1,s+1)}+L_1^{(s,s+2)}\right]=\frac{(2s+3)!}{3\alpha(s!)(s+1)!}\left\{\frac{1}{(s+2)}\sum_{i=0}^{s+2}\sum_{j=0}^{s+2}\sum_{l=0}^{t}\binom{s+2}{i}\binom{s+2}{j}\binom{s}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{2s-l+2}-\frac{2}{(s+1)}\sum_{i=0}^{s+1}\sum_{j=0}^{s+1}\sum_{l=0}^{s+1}\binom{s+1}{i}\binom{s+1}{j}\binom{s+1}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{2s-l+2}+\frac{1}{(s+2)}\sum_{i=0}^{s}\sum_{j=0}^{s}\sum_{l=0}^{s+2}\binom{s}{i}\binom{s}{j}\binom{s+2}{l}(-1)^i\lambda^{j+l}(1-\lambda)^{2s-l+2}\right\}\left\{\frac{(\lambda-1)e^{\theta(2s+i+j+l+3)}}{2s+i+j+l+3}Ei\left[-\theta(2s+i+j+l+3)\right]-\frac{2\lambda e^{\theta(2s+i+j+l+4)}}{2s+i+j+l+4}Ei\left[-\theta(2s+i+j+l+4)\right]\right\}, \tag{29}$$

Lower trimmed $(t=0)$:

In this case the r^{th} and the first three TL-moments are as follows:

$$L_r^{(s,0)}=\frac{1}{r}\sum_{k=0}^{r-1}\sum_{i=0}^{(r+s-k-1)}\sum_{j=0}^{(r+s-k-1)}\sum_{l=0}^{k}\frac{(-1)^{k+i}(1-\lambda)^{s-l}}{\alpha(r+s-k-1)!}\frac{\lambda^{j+l}(r+s)!}{!(k)!}\binom{r-1}{k}\binom{r+s-k-1}{i}\binom{r+s-k-1}{j}\binom{k}{l}\left\{\frac{(\lambda-1)e^{\theta(s+i+j+l+1)}}{s+i+j+l+1}Ei\left[-\theta(s+i+j+l+1)\right]-\frac{2\lambda e^{\theta(s+i+j+l+2)}}{s+i+j+l+2}Ei\left[-\theta(s+i+j+l+2)\right]\right\}, \tag{30}$$

$$L_1^{(s,0)} = \frac{(s+1)!}{\alpha(s!)} \sum_{i=0}^{s} \sum_{j=0}^{s} \binom{s}{i} \binom{s}{j} (-1)^i \lambda^j (1-\lambda)^{s-l+1} \left\{ \frac{\lambda-1}{s+i+j+1} e^{\theta(s+i+j+1)} Ei\left[-\theta(s+i+j+1)\right] - \frac{2\lambda}{s+i+j+2} e^{\theta(s+i+j+2)} Ei\left[-\theta(s+i+j+2)\right] \right\}, \tag{31}$$

$$L_2^{(s,0)} = \frac{1}{2}\left[L_1^{(s+1,0)} - L_1^{(s,1)}\right] = \frac{(s+2)!}{2\alpha s!} \left\{ \frac{1}{s+1} \sum_{i=0}^{s+1} \sum_{j=0}^{s+1} \binom{s+1}{i} \binom{s+1}{j} (-1)^i \lambda^j (1-\lambda)^{s-l+1} - \sum_{i=0}^{s} \sum_{j=0}^{s} \sum_{l=0}^{1} \binom{s}{i} \binom{s}{j} (-1)^i \lambda^{j+l} (1-\lambda)^{s-l+1} \right\} \left\{ \frac{(\lambda-1) e^{\theta(s+i+j+l+2)}}{s+i+j+l+2} Ei\left[-\theta(s+i+j+l+2)\right] - \frac{2\lambda e^{\theta(s+i+j+l+3)}}{s+i+j+l+3} Ei\left[-\theta(s+i+j+l+3)\right] \right\}, \tag{32}$$

and

$$L_3^{(s,0)} = \frac{1}{3}\left[L_1^{(s+2,0)} - 2L_1^{(s+1,1)} + L_1^{(s,2)}\right] = \frac{(s+3)!}{3\alpha s!} \left\{ \frac{\sum_{i=0}^{s+2} \sum_{j=0}^{s+2} \binom{s+2}{i} \binom{s+2}{j} (-1)^i \lambda^j (1-\lambda)^{s-l+2}}{(s+2)(s+1)} - \frac{2\sum_{i=0}^{s+1} \sum_{j=0}^{s+1} \sum_{l=0}^{1} \binom{s+1}{i} \binom{s+1}{j} (-1)^i \lambda^{j+l} (1-\lambda)^{s-l+2}}{(s+1)} + \frac{1}{2} \sum_{i=0}^{s} \sum_{j=0}^{s} \sum_{l=0}^{2} \binom{s}{i} \binom{s}{j} \binom{2}{l} (-1)^i \lambda^{j+l} (1-\lambda)^{s-l+2} \right\} \left\{ \frac{(\lambda-1) e^{\theta(s+i+j+l+3)}}{s+i+j+l+3} Ei\left[-\theta(s+i+j+l+3)\right] - \frac{2\lambda e^{\theta(s+i+j+l+4)}}{s+i+j+l+4} Ei\left[-\theta(s+i+j+l+4)\right] \right\}, \tag{33}$$

Upper trimmed $(s=0)$:

In this case the r^{th} and the first three TL-moments are as follows:

$$L_r^{(0,t)} = \frac{1}{r} \sum_{k=0}^{r-1} \sum_{i=0}^{(r-k-1)} \sum_{j=0}^{(r-k-1)} \sum_{l=0}^{t+k} \frac{(-1)^{k+i} \lambda^{j+l} (1-\lambda)^{t-l}}{\alpha(r-k-1)!} \frac{(r+t)!}{(t+k)!} \binom{r-1}{k} \binom{r-k-1}{i} \binom{r-k-1}{j} \binom{t+k}{l} \left\{ \frac{(\lambda-1) e^{\theta(t+i+j+l+1)}}{t+i+j+l+1} Ei\left[-\theta(t+i+j+l+1)\right] - \frac{2\lambda e^{\theta(t+i+j+l+2)}}{t+i+j+l+2} Ei\left[-\theta(t+i+j+l+2)\right] \right\}, \tag{34}$$

$$L_1^{(0,t)} = \sum_{l=0}^{t} \frac{\lambda^l (1-\lambda)^{t-l} (t+1)}{\alpha} \binom{t}{l} \left\{ \frac{\lambda-1}{t+l+1} e^{\theta(t+l+1)} Ei\left[-\theta(t+l+1)\right] - \frac{2\lambda}{t+l+2} e^{\theta(t+l+2)} Ei\left[-\theta(t+l+2)\right] \right\}, \tag{35}$$

$$L_2^{(0,t)} = \frac{1}{2}\left[L_1^{(1,t)} - L_1^{(0,t+1)}\right] = \frac{(t+2)!}{2\alpha t!} \left\{ \frac{1}{s+1} \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{l=0}^{t} \binom{t}{l} (-1)^i \lambda^{j+l} (1-\lambda)^{t-l+1} - \frac{1}{t+1} \sum_{l=0}^{t+1} \binom{t+1}{l} \lambda^l (1-\lambda)^{t-l+1} \right\} \left\{ \frac{(\lambda-1) e^{\theta(t+i+j+l+2)}}{t+i+j+l+2} Ei\left[-\theta(t+i+j+l+2)\right] - \frac{2\lambda e^{\theta(t+i+j+l+3)}}{t+i+j+l+3} Ei\left[-\theta(t+i+j+l+3)\right] \right\}, \tag{36}$$

and

$$L_3^{(0,t)} = \frac{1}{3}\left[L_1^{(2,t)} - 2L_1^{(1,t+1)} + L_1^{(0,t+2)}\right] = \frac{(t+3)!}{3\alpha t!}\left\{\frac{1}{2}\sum_{i=0}^{2}\sum_{j=0}^{2}\sum_{l=0}^{t}\binom{2}{i}\binom{2}{j}\binom{t}{l}(-1)^i \lambda^{j+l}(1-\lambda)^{t-l+2}\right.$$
$$\left.\frac{-2}{(t+1)}\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{l=0}^{t+1}\binom{1}{i}\binom{1}{j}\binom{t+1}{l}(-1)^i(1-\lambda)^{t-l+2}\lambda^{j+l} + \frac{1}{(t+2)(t+1)}\sum_{l=0}^{t+2}\binom{t+2}{l}\lambda^l(1-\lambda)^{t-l+2}\right\} \tag{37}$$
$$\left\{\frac{(\lambda-1)e^{\theta(t+i+j+l+3)}}{t+i+j+l+3}Ei\left[-\theta(t+i+j+l+3)\right] - \frac{2\lambda e^{\theta(t+i+j+l+4)}}{t+i+j+l+4}Ei\left[-\theta(t+i+j+l+4)\right]\right\},$$

7. Population L-Moments for TGD

In this section, we discuss the population L-moment of order r for the TGD as a special case of formula (22). Taking $s=t=0$ in (22), the population L-moment of order r for the TGD:

$$L_r = \sum_{k=0}^{r-1}\sum_{i=0}^{(r-k-1)}\sum_{j=0}^{(r-k-1)}\sum_{l=0}^{k}\frac{(-1)^{k+i}\lambda^{j+l}(r-1)!}{\alpha(1-\lambda)^l(r-k-1)!(k)!}\binom{r-1}{k}\binom{r-k-1}{i}\binom{r-k-1}{j}\binom{k}{l}$$
$$\left\{\frac{(\lambda-1)e^{\theta(i+j+l+1)}}{i+j+l+1}Ei\left[-\theta(i+j+l+1)\right] - \frac{2\lambda e^{\theta(i+j+l+2)}}{i+j+l+2}Ei\left[-\theta(i+j+l+2)\right]\right\}, \tag{38}$$

The first three L-moments can be obtained by taking $r=1,\ 2,\ 3$ in (46) as follows

$$L_1 = \frac{e^\theta}{\alpha}\left[(\lambda-1)Ei(-\theta) - \lambda e^\theta Ei(-2\theta)\right], \tag{39}$$

$$L_2 = \frac{e^\theta}{\alpha}\left\{(\lambda-1)Ei(-\theta) + \left(\lambda^2-3\lambda+1\right)e^\theta Ei(-2\theta) - 2\lambda(\lambda-1)e^{2\theta}Ei(-3\theta) + \lambda^2 e^{3\theta}Ei(-4\theta)\right\}, \tag{40}$$

and

$$L_3 = \frac{e^\theta}{\alpha}\left\{(1-\lambda)Ei(-\theta) + \frac{(\lambda-2)(5\lambda-3)}{2}e^\theta Ei(-2\theta) + \frac{6\lambda^3-16\lambda^2+10\lambda+2}{3}e^{2\theta}\right.$$
$$\left. Ei(-3\theta) - \frac{8\lambda^3-15\lambda^2+17\lambda}{4}e^{3\theta}Ei(-4\theta) + \frac{4\lambda^3+14\lambda^2+8\lambda}{5}e^{4\theta}Ei(-5\theta) - \frac{2\lambda^2(\lambda-2)}{3}e^{5\theta}Ei(-6\theta)\right\} \tag{41}$$

8. Application of TGD

In this section, we use a real data set to show that the TGD can be a better model than one based on the GD. We consider a data set of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so we have complete data with the exact times of failure. For previous studies with the data sets see Ref.[8] and Ref.[9]. These data are:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Table 1. *Estimated parameters of the TGD and GD.*

Model	Parameters Estimates	- L
TGD	$\hat{\alpha}=0.1879$, $\hat{\theta}=1.1480$, $\hat{\lambda}=0.8191$	64.25
GD	$\hat{\alpha}=0.1216$, $\hat{\theta}=3.3853$	87.203

The LR statistics to test the hypotheses $H_0:\lambda=0$ versus $H_1:\lambda\neq 0$: $\omega=45.906>6.635=\chi_1^2(\alpha=0.01)$, so we reject the null hypotheses.

9. Conclusion

In this article, we introduce a new generalization of the Gompertz distribution called transmuted Gompertz distribution and presented its theoretical properties. The estimation of parameters is approached by the method of maximum likelihood. We consider the likelihood ratio statistic to compare the model with its baseline model. An application of the transmuted Gompertz distribution to real data show that the new distribution can be used quite effectively to provide better than the Gompertz distribution.

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