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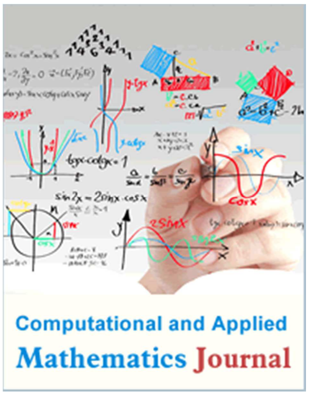

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Using Feedback Control Methods to Suppress a Modified Hyperchaotic Pan System

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Abstract
In this paper, we used four feedback control method to suppress a modified hyperchaotic Pan system to unstable equilibrium, and we found that the critical value for each method based on the Routh-Hurwitz theorem, we study the relationship between this value and asymptotically stable, unstable and Hopf Bifurcation. Finally, we found that the least complexity and cost of method depended only on the system's constants of critical value and do not depended on the method itself. Theoretical analysis, numerical simulation, illustrative examples and comparison are given to demonstrate the effectiveness of the proposed controllers.

1. Introduction

In recent year, chaos control and synchronization have been received more attention due to its potential applications to physics, chemical reactor, control theory, biological networks, artificial neural networks, telecommunications and secure communication[8]. Chaos control, in a broader sense, can be divided into two categories: one is to suppress the chaotic dynamical behavior and the other is to generate or enhance chaos in nonlinear systems [6].

Recently, after the pioneering work of Ott et al. [7,9,10,13,16], many different techniques and methods have been proposed to achieve chaos control, such as OGY method, time-delay feedback method, Lyapunov method , impulsive control method, sliding method control , differential geometric method, H_∞ control, adaptive control method, chaos suppression method, and so on. Among them, the feedback control is especially attractive and has been commonly applied to practical implementation due to its simplicity in configuration and implementation[3,6,7,16]. Generally speaking, there are two main approaches for controlling chaos: feedback control and non-feedback control. The feedback control approach offers many advantages such as robustness and computational complexity over the non- feedback control method[7,11,12,13].

Many attempts have been made to control hyperchaos and achieve synchronization of hyperchaotic systems [3,16]. Recently, Yan (2005) [8], Wang and Cai (2009) [16], Dou and Sun et al. (2009) [3], Zhu (2010)[14] and Zhuang et al. (2012)[15] suppressed hyperchaotic systems to unstable equilibrium by using feedback control method. In 2013, the Ref. [2] generated a new hyperchaotic based on Pan system [4,5] or Yang and Chen system via a state feedback controller which is called a modified hyperchaotic Pan system, which is described by the following mathematical model:

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz + w \\ \dot{z} = xy - bz \\ \dot{w} = -dy \end{cases} \tag{1}$$

where $(x, y, z, u) \in R^4$, and $a, b, c, d \in R$ are constant parameters, When parameters $a = 10$, $b = 8/3$ $c = 28$ and $d = 10$, system (1) is hyperchaotic and has two Lyapunov exponents, i.e. $LE_1 = 0.38352$, $LE_2 = 0.12714$ [2], The system (1) has only one equilibrium $O(0,0,0,0)$, and the equilibrium is an unstable under these parameters [2].

However, Some of the previous works founded that the coefficients of dislocate feedback control and enhancing feedback control were smaller than those of ordinary feedback control, and another some previous works founded that the coefficients of speed feedback control and enhancing feedback control were smaller than those of ordinary, but in this paper we found the main reason which make this coefficients are smaller than the rest of feedback control by depended on the system's constants of critical value, were each control method has critical value, and this value in ordinary feedback control depended on three system's constants i.e a,c and d, while in dislocate feedback control the critical value depended only one system's constants i.e a or c, and in speed feedback control depended on c or on a,c,d, Finally the enhancing feedback control depended on three system's constants i.e a,c and d.

2. Types of Feedback Control Method

2.1. Ordinary Feedback Control Method

For the ordinary feedback control, the system's variable is often multiplied by a coefficient as the feedback gain, and the feedback gain is added to the right -hand of the corresponding equation [7,11,12,13].

According to the above definition we have four forms (cases) to control the system (1), but after the test we found only one form (case) that effective on the stability of the system when it is a following form:

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz + w - ky \\ \dot{z} = xy - bz \\ \dot{w} = -dy \end{cases} \tag{2}$$

where k is the feedback coefficient and the critical value for this system is

$$k = k_{c(Ordinary)} = \frac{-(a^2+d-ac)+\sqrt{(a^2+d-ac)^2+4a^3c}}{2a} \tag{3}$$

Theorem 1. The solution of above system when a, b, c and $d > 0$ has the following cases:

(1) Asymptotically stable if $k > k_{c(Ordinary)}$.

(2) Unstable if $k < k_{c(Ordinary)}$.

(3) Hopf Bifurcation if $k = k_{c(Ordinary)}$.

Proof. The Jacobi matrix defined as:

$$J = \begin{bmatrix} -a & a & 0 & 0 \\ c-z & -k & -x & 1 \\ y & x & -b & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}_{O(0,0,0,0)} = \begin{bmatrix} -a & a & 0 & 0 \\ c & -k & 0 & 1 \\ 0 & 0 & -b & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}$$

and its characteristic equation is :

$$\lambda^4 + (k+a+b)\lambda^3 + (b(a+k) + ak - ac + d)\lambda^2 + (b(ak-ac+d) + ad)\lambda + abd = 0 \tag{4}$$

Solving equation (4) gives $\lambda_1 = -b$ and the following equation:

$$\lambda^3 + (k+a)\lambda^2 + (ak - ac + d)\lambda + ad = 0 \tag{5}$$

By using Routh-Hurwitz method, the equation (5) has all roots with negative real parts if and only if $A > 0$, $C > 0$ and $AB - C > 0$ where $A = k + a$, $B = ak - ac + d$ and $C = ad$, so , it is clear that $A, C > 0$ since $a, d > 0$ (given) and k is always positive and $AB > C$ it is possible under the condition $k > \frac{-(a^2+d-ac)+\sqrt{(a^2+d-ac)^2+4a^3c}}{2a} = k_{c(Ordinary)}$ therefore the system (2) is asymptotically stable if $k > k_{c(Ordinary)}$, unstable if $k < k_{c(Ordinary)}$ and Hopf Bifurcation if $k = k_{c(Ordinary)}$.

Proposition 1. Equation (4) has purely complex roots if and only if a, b, c and $d > 0$ and $k = k_{c(Ordinary)}$ In this case, the solution of equation (4) are $\lambda_1 = -b,\ \lambda_2 = -\frac{a^2-d+ac+\sqrt{E}}{2a}$, $\lambda_{3,4} = \pm i\sqrt{\frac{-a^2+d-ac+\sqrt{E}}{2}}$ where $E = (a^2+d-ac)^2 + 4a^3c$.

Proof. First get one root $\lambda_1 = -b$ from equation(4) then obtain cubic equation (equation 5) If $\lambda_{3,4} = \pm iu$ are the complex roots and λ_2 the real root of equation(4) then , form $\lambda_2 + \lambda_3 + \lambda_4 = -\frac{a^2-d+ac+\sqrt{E}}{2a}$. This easily leads to $\lambda_1 = -b,\ \lambda_2 = -\frac{a^2-d+ac+\sqrt{E}}{2a}$, $\lambda_{3,4} = \pm i\sqrt{\frac{-a^2+d-ac+\sqrt{E}}{2}}$ when $k = k_{c(Ordinary)}$.

Illustrative Example 1. Investigate for stability , unstable and Hopf Bifurcation of the following system:

$$\begin{cases} \dot{x} = 10(y-x) \\ \dot{y} = 28x - xz + w - ky \\ \dot{z} = xy - 8/3z \\ \dot{w} = -10y \end{cases} \quad (6)$$

Sol. $a = 10$, $b = 8/3$, $c = 28$ and $d = 10$, substitute this values in Eq(3) and Eq(4) we get the critical value $k = k_{c(Ordinary)} = 27.2683$ and the characteristic equation of system (6) is

$$(\lambda + 8/3)(\lambda^3 + (10+k)\lambda^2 + (10k-270)\lambda + 100) = 0 \quad (7)$$

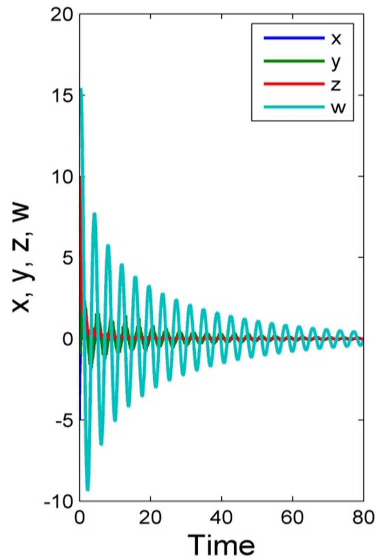
Now, according to theorem 1 the roots of equation (7) depended of the value of k as following:

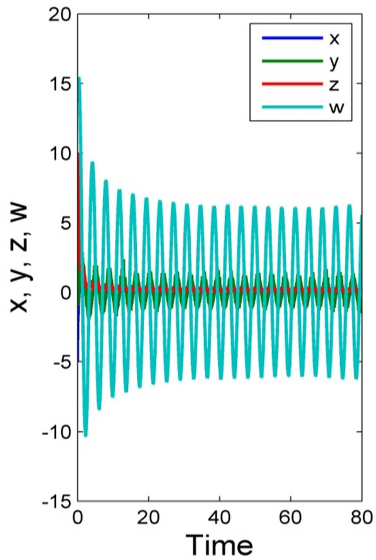
If $k = 27.5$ the equation (7) became $(\lambda + 8/3)(\lambda^3 + 37.5\lambda^2 + 5\lambda + 100) = 0$, and the all roots have negative real parts $\lambda_1 = -8/3$, $\lambda_2 = -37.4378$, $\lambda_{3,4} = -0.0311 \mp i1.6341$ therefore the system(6) is asymptotically stable.

If $k = 27$ the equation (7) became $(\lambda + 8/3)(\lambda^3 + 37\lambda^2 + 100) = 0$, then not all roots have negative real parts $\lambda_1 = -8/3$, $\lambda_2 = -37.0728$, $\lambda_{3,4} = 0.0364 \mp i1.6420$ therefore the system(6) is unstable.

Finally, if $k = 27.2683$ the equation(7) became $(\lambda + 8/3)(\lambda^3 + 37.2683\lambda^2 + 2.683\lambda + 100) = 0$, we have two roots with negative real parts $\lambda_1 = -8/3$, $\lambda_2 = -37.2683$ and the other roots have purely complex $\lambda_{3,4} = \pm i1.6380$ therefore the system(6) is Hopf Bifurcation.

Also we can justified the same result that obtain in theorem1 by using MATLAB program where numerical simulations are used to investigate the controlled hyperchaotic system (6) using fourth-order Runge-Kutta scheme, The feedback coefficients are given by $k = 27.5$, $k = 27$ in figure 1 a, b respectively. The behaviors of the states $x(t), y(t), z(t), w(t)$ of the hyperchaotic system (6) show converging to $O(0,0,0,0)$ when $k = 27.5$ (Fig1,a.) and divergence to $O(0,0,0,0)$ when $k = 27$ (Fig1,b.)





(a) System (6) converge to O(0,0,0,0) where k=27.5 (b) System (6) divergence to O(0,0,0,0) where k=27

Fig. 1. *The difference of the state of the controlled system (6) with the control gain change.*

We can briefly describe illustrative example 1 by the following table (Table 1) also we can applied another values in theorem 1 by the same table.

Table 1. *Relationship between critical value and asymptotically stable, unstable and Hopf Bifurcation for system (2).*

Input values	Compute k_c	Compare	Roots	State
		k =27.5	$\lambda_2 = -37.4378$, $\lambda_{3,4} = -0.0311 \mp i1.6341$	asymptotically stable
a=10, c=28, d=10	27.26832	k=27	$\lambda_2 = -37.0728$, $\lambda_{3,4} = 0.0364 \mp i1.6420$	unstable
		k=27.2683	$\lambda_2 = -37.2683$, $\lambda_{3,4} = \pm i1.6381$	Hopf Bifurcation
		k =35	$\lambda_2 = -69.9429$, $\lambda_{3,4} = -0.0286 \mp i2.0006$	asymptotically stable
a=35, c=35, d=8	34.8859	k=34	$\lambda_2 = -69.4468$, $\lambda_{3,4} = 0.2234 \mp i1.9955$	unstable
		k=34.8859	$\lambda_2 = -69.8859$, $\lambda_{3,4} = \pm i2.0016$	Hopf Bifurcation

Corollary 1. If the ordinary feedback control is the form as

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz + w \\ \dot{z} = xy - bz \\ \dot{w} = -dy - kw \end{cases} \tag{8}$$

where k is the feedback coefficient and the critical value for this system is

$$k = k_{c(Ordinary)} = \frac{-(a^2+d)+\sqrt{(a^2+d)^2+4a^3c}}{2a} \tag{9}$$

and k is a positive but this form, the system cannot converge to origin point .

2.2. Dislocated Feedback Control Method

If a system variable multiplied by a coefficient is added to the right -hand of another equation, then this method is called dislocated feedback control[7,11,12,13,14,15].

By above definition we have twelve forms (cases) to control the system (1), but after the test we found only two forms (cases) that effective on the stability of the system when it is a following form :

$$\begin{cases} \dot{x} = a(y-x) - ky \\ \dot{y} = cx - xz + w \\ \dot{z} = xy - bz \\ \dot{w} = -dy \end{cases} \tag{10}$$

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz + w - kx \\ \dot{z} = xy - bz \\ \dot{w} = -dy \end{cases} \tag{11}$$

where k is the feedback coefficient and the critical value for system (10) and system (11) respectively are:

$$k = k_{c(Dislocated)} = a \tag{12}$$

$$k = k_{c(Dislocated)} = c \tag{13}$$

Theorem. 2 (i) The solution of system (10) when a, b, c and $d > 0$ has the following cases:

(1) Asymptotically stable if $k > a$
(2) Unstable if $k < a$
(3) Hopf Bifurcation if $k = a$

(ii) The solution of system (11) when a, b, c and $d > 0$ has the following cases:

(1) Asymptotically stable if $k > c$
(2) Unstable if $k < c$
(3) Hopf Bifurcation if $k = c$

Proof. (i) The Jacobi matrix defined as:

$$J = \begin{bmatrix} -a & a-k & 0 & 0 \\ c-z & 0 & -x & 1 \\ y & x & -b & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}_{O(0,0,0,0)} = \begin{bmatrix} -a & a-k & 0 & 0 \\ c & 0 & 0 & 1 \\ 0 & 0 & -b & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}$$

and its characteristic equation is :

$$\lambda^4 + (a+b)\lambda^3 + (ab + ck + d - ac)\lambda^2 + (b(ck + d - ac) + ad)\lambda + abd = 0 \tag{14}$$

Solving equation (14) gives $\lambda_1 = -b$ and the following equation:

$$\lambda^3 + a\lambda^2 + (ck + d - ac)\lambda + ad = 0 \tag{15}$$

By using Routh-Hurwitz method, the equation (15) has all roots with negative real parts if and only if $A > 0$, $C > 0$ and $AB - C > 0$ where $A = a$, $B = ck + d - ac$ and $C = ad$, so , it is clear that $A, C > 0$ since $a, d > 0$ (given) and since k is always positive then $AB > C$ is satisfied under the condition $k > a = k_{c(Dislocated)}$ therefore the system (10) is asymptotically stable if $k > a$, unstable if $k < a$ and Hopf Bifurcation if $k = a$.

Proof. (ii) Analogously as in above proof .

Proposition2. Equation (14) has purely complex roots if and only if a, b, c and $d > 0$ and $k = k_{c(Dislocated)} = a$. In this case, the solution of equation (14) are $\lambda_1 = -b,\ \lambda_2 = -a$, $\lambda_{3,4} = \pm i\sqrt{d}$.

Proof. First get one root $\lambda_1 = -b$ from equation(14) then obtain cubic equation (equation 15) If $\lambda_{3,4} = \pm iu$ are the complex roots and λ_2 the real root of equation(14) then , form $\lambda_2 + \lambda_3 + \lambda_4 = -a$. This easily leads to $\lambda_1 = -b,\ \lambda_2 = -a$, $\lambda_{3,4} = \pm i\sqrt{d}$.

Corollary 2. System (11) has the same roots of system (10) when $k = c = k_{c(Dislocated)}$, General system (10) and system (11) have the same roots when $k = k_{c(Dislocated)}$.

We can briefly describe theorem 2 by the following table where Table 2 for system (10) and Table 3 for system (11).

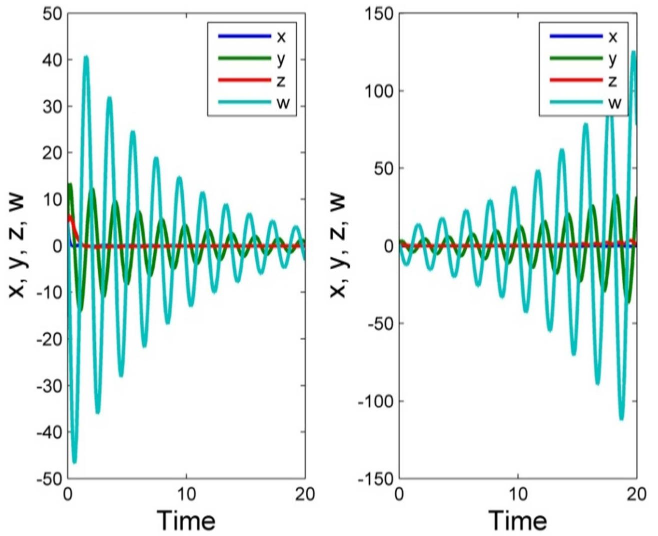
Table 2. *Relationship between critical value and asymptotically stable, unstable and Hopf Bifurcation for system (10).*

Input values	Compute k_c	Compare	Roots	State
		k =10.1	$\lambda_2=-9.7399$, $\lambda_{3,4}=-0.1300\mp i3.2016$	asymptotically stable
a=10, c=28, d=10	10	k=9.9	$\lambda_2=-10.2494$, $\lambda_{3,4}=0.1247\mp i3.1211$	unstable
		k=10	$\lambda_2=-10$, $\lambda_{3,4}=\pm i3.1623$	Hopf Bifurcation
		k =36	$\lambda_2=-33.9770$, $\lambda_{3,4}=-0.5115\mp i2.8284$	asymptotically stable
a=35, c=35, d=8	35	k=34	$\lambda_2=-35.9671$, $\lambda_{3,4}=0.4836\mp i2.7479$	unstable
		k=35	$\lambda_2=-35$, $\lambda_{3,4}=\pm i2.8284$	Hopf Bifurcation

Table 3. *Relationship between critical value and asymptotically stable, unstable and Hopf Bifurcation for system (11).*

Input values	Compute k_c	Compare	Roots	State
		k =28.1	$\lambda_2=-9.9084$, $\lambda_{3,4}=-0.0458\mp i3.1765$	asymptotically stable
a=10, c=28, d=10	28	k=27	$\lambda_2=-10.8495$, $\lambda_{3,4}=0.4248\mp i3.0061$	unstable
		k=28	$\lambda_2=-10$, $\lambda_{3,4}=\pm i3.1623$	Hopf Bifurcation
		k =35.1	$\lambda_2=-34.9004$, $\lambda_{3,4}=-0.0498\mp i2.8320$	asymptotically stable
a=35, c=35, d=8	35	k=34.9	$\lambda_2=-35.0991$, $\lambda_{3,4}=0.0495\mp i2.8240$	unstable
		k=35	$\lambda_2=-35$, $\lambda_{3,4}=\pm i2.8284$	Hopf Bifurcation

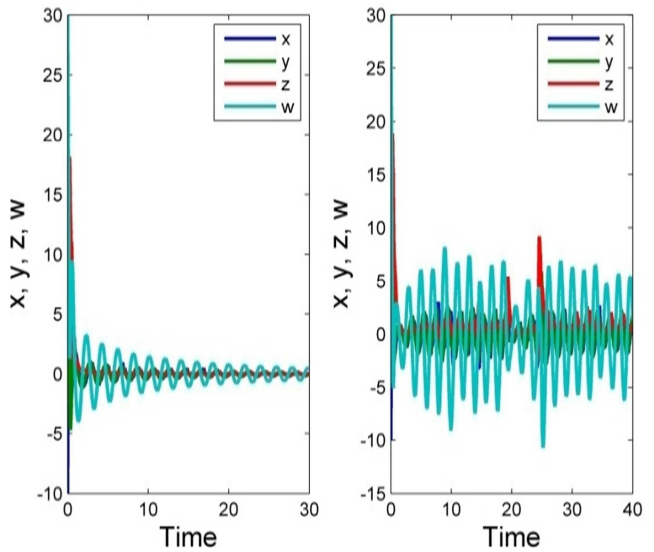
we noted from two tables the corollary 2 is satisfied when $k=k_{c(Dislocated)}$. and Fig. 2- a, b show converging and divergence to origin point when $k=10.1$, $k=9.9$ respectively for system (10)



(a) System (10) converge to O(0,0,0,0) where k=10.1 (b) System (10) divergence to O(0,0,0,0) where k=9.9

Fig. 2. *The difference of the state of the controlled system (10) with the control gain change.*

and Fig. 3- a, b show converging and divergence to origin point when $k=28.1$, $k=27$ respectively for system (11) when $a=10, c=28, d=10$.



(a) System (11) converge to O(0,0,0,0) where k=28.1 (b) System (11) divergence to O(0,0,0,0) where k=27

Fig. 3. *The difference of the state of the controlled system (11) with the control gain change.*

Corollary 3. If the dislocated feedback control is the form as

$$\begin{cases} \dot{x}=a(y-x)-kw \\ \dot{y}=cx-xz+w \\ \dot{z}=xy-bz \\ \dot{w}=-dy \end{cases} \tag{16}$$

where k is the feedback coefficient and the critical value for this system is

$$k = k_{c(Dislocated)} = \frac{a^2}{d} \tag{17}$$

and k is a positive but this form, the system cannot converge to origin point .

2.3. Speed Feedback Control Method

For the speed feedback control, the independent variable of a system function is often multiplied by a coefficient as the feedback gain, so the method is called displacement feedback control. Similarly, if the derivative of an independent variable is multiplied by a coefficient as the feedback gain, it is called speed feedback control[3,6,7,8,11,12,13,14,16].

By above definition we have twelve forms (cases) to control the system (1), but after the test we found only two forms (cases) that effective on the stability of the system when it is a following form :

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz + w \\ \dot{z} = xy - bz \\ \dot{w} = -dy - k\dot{x} \end{cases} \tag{18}$$

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz + w \\ \dot{z} = xy - bz \\ \dot{w} = -dy - k\dot{y} \end{cases} \tag{19}$$

where k is the feedback coefficient and the critical value for system (18) and system (19) respectively are:

$$k = k_{c(Speed)} = c \tag{20}$$

$$k = k_{c(Speed)} = \frac{-(a^2+d-ac)+\sqrt{(a^2+d-ac)^2+4a^3c}}{2a} \tag{21}$$

Theorem.3 (i) The solution of system (18) when a,b,c and $d>0$ has the following cases:

(1) Asymptotically stable if $k > c$

(2) Unstable if $k < c$

(3) Hopf Bifurcation if $k = c$

(ii) The solution of system (19) when a,b,c and $d>0$ has the following cases:

(1) Asymptotically stable if $k > k_{c(Speed)}$

(2) Unstable if $k < k_{c(Speed)}$

(3) Hopf Bifurcation if $k = k_{c(Speed)}$

Proof. (i) The Jacobi matrix defined as:

$$J = \begin{bmatrix} -a & a & 0 & 0 \\ c-z & 0 & -x & 1 \\ y & x & -b & 0 \\ ak & -d-ak & 0 & 0 \end{bmatrix}_{O(0,0,0,0)} = \begin{bmatrix} -a & a & 0 & 0 \\ c & 0 & 0 & 1 \\ 0 & 0 & -b & 0 \\ ak & -d-ak & 0 & 0 \end{bmatrix}$$

and its characteristic equation is :

$$\lambda^4 + (a+b)\lambda^3 + (ak+d-ac+ab)\lambda^2 + (b(ak+d-ac)+ad)\lambda + abd = 0 \tag{22}$$

Solving equation (22) gives $\lambda_1 = -b$ and the following equation:

$$\lambda^3 + a\lambda^2 + (ak+d-ac)\lambda + ad = 0 \tag{23}$$

By using Routh-Hurwitz method, the equation (23) has all roots with negative real parts if and only if $A>0,\ C>0$ and $AB-C>0$ where $A=a$, $B=ak+d-ac$ and $C=ad$, so , it is clear that $A,C>0$ since $a,d>0$ (given) and since k is always positive then $AB>C$ is satisfied under the condition $k > ac = k_{c(Speed)}$ therefore the system (18) is asymptotically stable if $k>c$, unstable if $k<c$ and Hopf Bifurcation if $k=c$.

Proof. (ii) Analogously as in above proof .

Proposition3. Equation (22) has purely complex roots if and only if a,b,c and $d>0$ and $k = k_{c(Speed)} = c$. In this case, the solution of equation (22) are $\lambda_1 = -b,\ \lambda_2 = -a$, $\lambda_{3,4} = \pm i\sqrt{d}$.

Corollary 4. System (18) has the same roots of system (10) and system(11) under the condition $k = c = k_{c(Speed)}$, General system (10) ,system (11) and system(18) have the same roots when $k = k_c$.

Corollary 5. System (11) and system(18) have the same roots without any condition.

Corollary 6. System (2) and system(19) have the same roots without any condition.

We can briefly describe theorem 3 by the following table where Table 4 for system (18) and Table 5 for system (19).

Table 4. *Relationship between critical value and asymptotically stable, unstable and Hopf Bifurcation for system (18).*

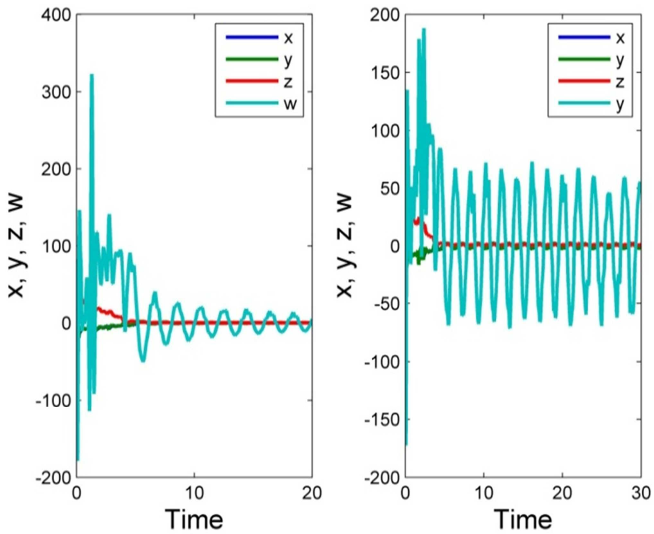
Input values	Compute k_c	Compare	Roots	State
		k =28.1	$\lambda_2 = -9.9084$, $\lambda_{3,4} = -0.0458 \mp i3.1765$	asymptotically stable
a=10, c=28, d=10	28	k=27	$\lambda_2 = -10.8495$, $\lambda_{3,4} = 0.4248 \mp i3.0061$	unstable
		k=28	$\lambda_2 = -10$, $\lambda_{3,4} = \pm i3.1623$	Hopf Bifurcation
		k =35.1	$\lambda_2 = -34.9004$, $\lambda_{3,4} = -0.0498 \mp i2.8320$	asymptotically stable
a=35, c=35, d=8	35	k=34.9	$\lambda_2 = -35.0991$, $\lambda_{3,4} = 0.0495 \mp i2.8240$	unstable
		k=35	$\lambda_2 = -35$, $\lambda_{3,4} = \pm i2.8284$	Hopf Bifurcation

we noted from table2, table3 and table4 the corollary 4 is satisfied under the condition $k = k_c$. also if we compare table4 for system(18) with table3 for system(11) we found the same results, therefore the corollary 5 is satisfied.

Table 5. *Relationship between critical value and asymptotically stable, unstable and Hopf Bifurcation for system (19).*

Input values	Compute k_c	Compare	Roots	State
		k =27.5	$\lambda_2 = -37.4378$, $\lambda_{3,4} = -0.0311 \mp i1.6341$	asymptotically stable
a=10, c=28, d=10	27.26832	k=27	$\lambda_2 = -37.0728$, $\lambda_{3,4} = 0.0364 \mp i1.6420$	unstable
		k=27.2683	$\lambda_2 = -37.2683$, $\lambda_{3,4} = \pm i1.6381$	Hopf Bifurcation
		k =35	$\lambda_2 = -69.9429$, $\lambda_{3,4} = -0.0286 \mp i2.0006$	asymptotically stable
a=35, c=35, d=8	34.8859	k=34	$\lambda_2 = -69.4468$, $\lambda_{3,4} = 0.2234 \mp i1.9955$	unstable
		k=34.8859	$\lambda_2 = -69.8859$, $\lambda_{3,4} = \pm i2.0016$	Hopf Bifurcation

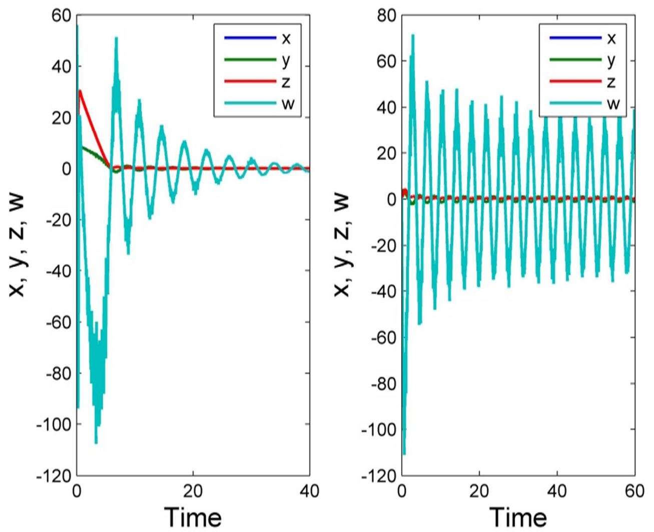
we noted from table5 and table1 have the same results. so, the corollary 6 is satisfied. and Fig. 4- a, b show converging and divergence to origin point when $k = 28.1$, $k = 27$ respectively for system (18).



(a) System (18) converge to O(0,0,0,0) where k=28.1 (b) System (18) divergence to O(0,0,0,0) where k=27

Fig. 4. *The difference of the state of the controlled system (18) with the control gain change.*

and Fig. 5- a, b show converging and divergence to origin point when $k = 27.5$, $k = 27$ respectively for system (19) when $a = 10, c = 28, d = 10$.



(a) System (19) converge to O(0,0,0,0) where k=27.5 (b) System (19) divergence to O(0,0,0,0) where k=27

Fig. 5. *The difference of the state of the controlled system (19) with the control gain change.*

Corollary 7. If the speed feedback control is the form as

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = cx - xz + w - k\dot{w} \\ \dot{z} = xy - bz \\ \dot{w} = -dy \end{cases} \tag{24}$$

where k is the feedback coefficient and the critical value for this system is

$$k = k_{c(Speed)} = \frac{-(acd - d^2 - a^2 d) + \sqrt{(acd - d^2 - a^2 d)^2 + 4a^3 cd^2}}{2ad^2} \tag{25}$$

and k is a positive but this form, the system cannot converge to origin point .

2.4. Enhancing Feedback Control Method

It is difficult for a complex system to be controlled by only one feedback variable, and in such cases the feedback gain is always very large. So we consider using multiple variables multiplied by a proper coefficient as the feedback gain. This method is called enhancing feedback control [7,11,12,13 - 15] .

By above definition we have eleven forms (cases) to control the system (1), but after the test we found three forms (cases) that effective on the stability of the system when it is a following form :

$$\begin{cases} \dot{x} = a(y-x) - kx \\ \dot{y} = cx - xz + w - ky \\ \dot{z} = xy - bz \\ \dot{w} = -dy \end{cases} \tag{26}$$

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz + w - ky \\ \dot{z} = xy - bz - kz \\ \dot{w} = -dy \end{cases} \tag{27}$$

$$\begin{cases} \dot{x} = a(y-x) - kx \\ \dot{y} = cx - xz + w - ky \\ \dot{z} = xy - bz - kz \\ \dot{w} = -dy \end{cases} \tag{28}$$

where k is the feedback coefficient.

Proposition4. The critical value for system (26) depended of discriminate (Δ) of cubic equation as:

$$k = k_{c(Enhancing)} = \sqrt[6]{16(q^2 - \Delta)}\cos\frac{\cos^{-1}\frac{-q}{\sqrt{q^2-\Delta}}}{3} - \frac{a}{3} \quad if \quad \Delta < 0 \tag{29}$$

$$k = k_{c(Enhancing)} = \sqrt[3]{\frac{q}{2}} - \frac{a}{3} \quad if \;\; \Delta = 0 \tag{30}$$

$$k = k_{c(Enhancing)} = \sqrt[3]{\frac{-q-\sqrt{\Delta}}{2}} + \sqrt[3]{\frac{-q+\sqrt{\Delta}}{2}} - \frac{a}{3} \quad if \;\; \Delta > 0 \tag{31}$$

where q and Δ for system (26) are defined as:

$$q = -\frac{1}{4}ad \quad , \qquad \Delta = \frac{1}{4}a^4c^2 + \frac{1}{54}(d - 2ac + a^2)^3 + \frac{1}{4}a^3c(d - 2ac + a^2) - \frac{1}{48}a^2(d - 2ac + a^2) - \frac{1}{4}a^5c$$

Proof. The Jacobi matrix of system (26) is

$$J = \begin{bmatrix} -a & a & 0 & 0 \\ c-z & 0 & -x & 1 \\ y & x & -b & 0 \\ ak & -d-ak & 0 & 0 \end{bmatrix}_{O(0,0,0,0)} = \begin{bmatrix} -10-k & 10 & 0 & 0 \\ 28 & -k & 0 & 1 \\ 0 & 0 & -8/3 & 0 \\ 0 & -10 & 0 & 0 \end{bmatrix} \tag{32}$$

and its characteristic equation is :

$$\lambda^4 + (2k + a + b)\lambda^3 + (b(2k + a) + k^2 + ak + d - ac)\lambda^2 + (b(k^2 + ak + d - ac) - d(k + a))\lambda + bd(k + a) = 0 \tag{33}$$

Solving equation (33) gives $\lambda_1 = -b$ and the following equation:

$$\lambda^3 + (2k + a)\lambda^2 + (k^2 + ak + d - ac)\lambda + d(k + a) = 0 \tag{34}$$

we applied the condition $AB = C$ if $A, B, C > 0,$ on Eq (34) we get the following equation:

$$k^3+\frac{3}{2}ak^2+\frac{1}{2}(d-2ac+a^2)k-\frac{1}{2}a^2c=0 \tag{35}$$

Now, Eq(35) is cubic equation, To solve this equation with respective to k we use Gardan method which exist in Ref [1] consequently, we get the critical value for system (26) depended of Δ in Eq(29), Eq(30)and Eq(31) The proof is completed.

Theorem.4 The solution of system (26) when a,b,c and $d>0$ has the following cases:

(1) Asymptotically stable if $k>k_{c(Enhancing)}$

(2) Unstable if $k<k_{c(Enhancing)}$

(3) Hopf Bifurcation if $k=k_{c(Enhancing)}$

Proof. Analogously as in proof in previous theorem.

Corollary 8. System (27) has the same critical value and roots of system (2) without any condition, also system (28) has the same critical value and roots of system (26) without any condition.

We can briefly describe theorem 4 by the following table (Table 6)

Table 6. *Relationship between critical value and asymptotically stable, unstable and Hopf Bifurcation for system (26).*

Input values	Compute k_c	Compare	Roots	State
		k =12.3630	$\lambda_2=-34.7250$, $\lambda_{3,4}=-0.0005\mp i\,2.5377$	asymptotically stable
a=10, c=28, d=10	12.3620	k=12.3610	λ_2=−34.7230, $\lambda_{3,4}=$ 0.0005$\mp i\,2.5377$	unstable
		k=12.3620	$\lambda_2=-34.7240$, $\lambda_{3,4}=\pm i\,2.5377$	Hopf Bifurcation
		k =21.7	$\lambda_2=-78.3030$, $\lambda_{3,4}=-0.0485\mp i\,2.4064$	asymptotically stable
a=35, c=35, d=8	21.6029	k=21.5	λ_2=−78.1029, $\lambda_{3,4}$ =0.0515$\mp i\,2.4051$	unstable
		21.6029	$\lambda_2=-78.2058$, $\lambda_{3,4}=\pm i\,2.4063$	Hopf Bifurcation

Corollary 9. If the enhancing feedback control are the forms as:

$$\begin{cases}\dot{x}=a(y-x)-kx\\ \dot{y}=cx-xz+w\\ \dot{z}=xy-bz\\ \dot{w}=-dy-kw\end{cases} \tag{36}$$

$$\begin{cases}\dot{x}=a(y-x)\\ \dot{y}=cx-xz+w-ky\\ \dot{z}=xy-bz\\ \dot{w}=-dy-kw\end{cases} \tag{37}$$

$$\begin{cases}\dot{x}=a(y-x)\\ \dot{y}=cx-xz+w\\ \dot{z}=xy-bz-kz\\ \dot{w}=-dy-kw\end{cases} \tag{38}$$

$$\begin{cases}\dot{x}=a(y-x)-kx\\ \dot{y}=cx-xz+w\\ \dot{z}=xy-bz-kz\\ \dot{w}=-dy-kw\end{cases} \tag{39}$$

$$\begin{cases}\dot{x}=a(y-x)\\ \dot{y}=cx-xz+w-ky\\ \dot{z}=xy-bz-kz\\ \dot{w}=-dy-kw\end{cases} \tag{40}$$

$$\begin{cases}\dot{x}=a(y-x)-kx\\ \dot{y}=cx-xz+w-ky\\ \dot{z}=xy-bz\\ \dot{w}=-dy-kw\end{cases} \tag{41}$$

$$\begin{cases}\dot{x}=a(y-x)-kx\\ \dot{y}=cx-xz+w-ky\\ \dot{z}=xy-bz-kz\\ \dot{w}=-dy-kw\end{cases} \tag{42}$$

and k is a positive but this forms, the systems cannot converge to origin point based on critical value .

Corollary 10. Systems(37,40,41 and 42) can be suppred by using Routh-Hurwitz method.

3. Comparison

In this section, we compare between four feedback control method to suppress hyperchaos system (1) to unstable equilibrium, and we found the difference between them, and the following table explain this difference .

Table 7. *Difference between four feedback control method for system (1).*

	Ordinary feedback control	Dislocated feedback control	Speed feedback control	Enhancing feedback control
1	Contain four cases	Contain twelve cases	Contain twelve cases	Contain eleven cases
2	Only one case effective on the system	Only two case effective on the system	Only two case effective on the system	Seven cases effective on the system
3	Depended on three constants	Depended only one constant	first case depended only one constant and	Seven case depended on three

	Ordinary feedback control	Dislocated feedback control	Speed feedback control	Enhancing feedback control
		for each case	second case depended on three constants	constants
4	Linear feedback control	Linear feedback control	first case is Linear feedback control and second case is non-linear feedback control	Linear feedback control
5	Depended on critical value	Depended on critical value	Depended on critical value	Depended on critical value and Routh-Hurwitz

4. Conclusions

In this paper, the control problem of a modified hyperchaotic Pan system is investigated: ordinary feedback control, dislocated feedback control , speed feedback control , enhancing feedback control are used to suppress hyperchaos to unstable equilibrium. we noted that the dislocated feedback control is more simplicity to compute the effective value (critical value) that reduced the cost from the another methods, since the coefficients of this method depended only one system's constant (a or c), while the coefficients of speed feedback control depended on one system's constant (a) or three system's constant (a,c,d)), Finally both ordinary and enhancing feedback control are more complicity to compute the effective value (critical value) that reduced the cost, since the coefficients of this methods depended on three system's constant (a,c,d), also we noted that system's constant b do not effective on these methods.

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