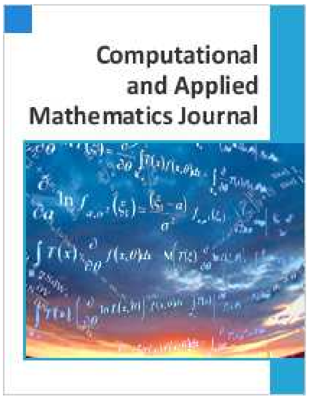
Computational and Applied Mathematics Journal
2015; 1(5): 393-400
Published online October 20, 2015 (http://www.aascit.org/journal/camj)
ISSN: 2381-1218 (Print); ISSN: 2381-1226 (Online)

American Association for Science and Technology



Keywords
Almost Sure Permanence,
Epidemic Model,
Brownian Motion,
Sir,
ItˆO Formula

Received: May 19, 2015
Revised: June 11, 2015
Accepted: June 12, 2015

Almost Sure Permanence of Stochastic SIR Epidemic Model

Shaobin Rao[1], Xiaorong Gan[1, 2]

[1]Department of Applied Mathematics, Kunming University of Science and Technology, Kunming, Yunnan, China
[2]Kunming University of Science and Technology, City College, Kunming, China

Email address
627097605@qq.com (X. R. Gan)

Citation
Shaobin Rao, Xiaorong Gan. Almost Sure Permanence of Stochastic SIR Epidemic Model. *Computational and Applied Mathematics Journal.* Vol. 1, No. 5, 2015, pp. 393-400.

Abstract
In this paper, we consider a non-autonomous stochastic SIR epidemic model. Some new sufficient conditions which guarantee the permanence of the stochastic epidemic model are obtained. The results in this paper imply that the intensity of white noise has no effect on the permanence of the infective and the removed class of system.

1. Introduction

Epidemiology is the branch of biology which deals with the mathematical modeling of spread of diseases, many problems arising in epidemiology may be described, in a first formulation, by means of differential equations, this means that the models are constructed by averaging some population and keeping only the time variable. To the best of our knowledge the first mathematical model of epidemiology was formulated and solved by Daniel Bernoulli in 1760. Since the time of Kermack and Mckendrick [5], the study of mathematical epidemiology has grown rapidly, with a large variety of models having been formulated and applied to infectious diseases[2, 3, 11]. Consider a population which remains constant and which is divide into three classes: the susceptible, denoted by S, who can catch the disease; the infective, denoted by I, who are infected and can transmit the disease to the susceptible, and the removed class, denoted by R, who had the disease and recovered or died or have developed immunity or have been removed from contact with the other classes. Since from the modeling perspective only the overall state of a person with respect to the disease is relevant, the progress of individuals is schematically described by $S \to I \to R$.

These types of models are known as SIR models. In recent years, many scholars pay extensive attention to the dynamic behaviours of SIR epidemic models. We refer the readers to [1, 6, 9, 10, 12] In [1], Bai and Zhou formulated a non-autonomous SIR epidemic model with saturated incidence rate and constant removal rate by introducing the periodic transmission rate $\beta(t)$ as follows:

$$\begin{cases} \dot{S} = \alpha - \mu S(t) - \dfrac{\beta(t)S(t)I(t)}{k_1 + k_2 I(t)}, \\ \dot{I} = \dfrac{\beta(t)S(t)I(t)}{k_1 + k_2 I(t)} - (\mu+\gamma)I(t) - h(I), \\ \dot{R} = \gamma I(t) + h(I) - \mu R(t), \end{cases} \tag{1.1}$$

where α is the recruitment rate, μ is the natural death rate, γ is the recovery rate of the infective, $\beta(t)$ is the transmission rate at time t and h is a treatment function, which is a positive constant σ for $I > 0$, and zero for $I = 0$, and $k_1 > 0$, $k_2 > 0$, however, the growth rate in the biology species system should exhibit random fluctuation[8]. Assume that the growth rate r is perturbed by environmental noise with $r \to r + \sigma_i \dot{B}(t)$, as we all know, the recruitment rate, natural death rate and the recovery rate of the infective are not constant but vary with t by utilizing the case, system (1.1) have the reasonable and important stochastic model as follows:

$$\begin{cases} dS(t) = [\alpha(t) - \mu(t)S(t) - \dfrac{\beta(t)S(t)I(t)}{k_1 + k_2 I(t)}]dt - \sigma_1 S(t)dB_1(t), \\ dI(t) = [\dfrac{\beta(t)S(t)I(t)}{k_1 + k_2 I(t)} - (\mu(t) + \gamma(t))I(t) - h(I)]dt - \sigma_2 I(t)dB_2(t), \\ dR(t) = [\gamma(t)I(t) + h(I) - \mu(t)R(t)]dt - \sigma_3 R(t)dB_3(t), \end{cases} \tag{1.2}$$

Where $B_1(t)$, $B_2(t)$, $B_3(t)$ are independent standard Brownian motions, σ_1^2, σ_2^2, σ_3^2 are the intensity of white noise, $\beta(t)$, $\alpha(t)$, $\mu(t)$ and $\gamma(t)$ are bounded and continuous functions, $\beta(t) > 0$, $\alpha(t) > 0$, $\mu(t) > 0$ and $\gamma(t) > 0$ on $t \in [0, +\infty)$ It is well known that the permanence is a very important and interesting topic in mathematical ecology, which means that a biology species system will survive forever. In general, a deterministic species system is permanent, if system has the following properties

$$0 < N \le \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) \le M < \infty, \quad i = 1, 2, \ldots n. \tag{1.3}$$

Lots of results about permanence have been obtained for deterministic population models. As far as stochastic population models are concerned, it is natural and reasonable to consider their permanence. Throughout this paper, we set

$$\mu^+ = \sup_{t\in(0,\infty)} \mu(t) \quad \mu^- = \inf_{t\in(0,\infty)} \mu(t) \quad \alpha^+ = \sup_{t\in(0,\infty)} \alpha(t) \quad \alpha^- = \inf_{t\in(0,\infty)} \alpha(t)$$

$$\beta^+ = \sup_{t\in(0,\infty)} \beta(t) \quad \beta^- = \inf_{t\in(0,\infty)} \beta(t) \quad \gamma^+ = \sup_{t\in(0,\infty)} \gamma(t) \quad \gamma^- = \inf_{t\in(0,\infty)} \gamma(t)$$

The remaining part of this paper is organized as follows: In Section 2, we will state several definitions and lemmas which will be useful in the proving of main results of this paper. In Section 3, we obtain some new sufficient conditions for the permanence of system (1.2).

2. Preliminaries

Definition 2.1. ([7]) A stochastic system is said to be almost surely stochastically permanent if for any initial value $x_0 \in R_+^n$, the solution $x(t) = (x_1(t), x_2(t), ..., x_n(t),)$, satisfies

$$0 < N < \liminf_{t\to\infty} x_i(t) \le \limsup_{t\to\infty} x_i(t) < M < \infty, \quad a.s. i = 1, 2, ..., n$$

Lemma 2.1. ([4]) Assume that X(t) and Y (t) are Itˆo process, then we have

$$d[X(t)Y(t)] = X(t)dY(t) + Y(t)dX(t) + dX(t).dY(t).$$

Lemma 2.2. Consider the linear stochastic differential equation

$$dx(t) + r(t)x(t)dt + \sigma x(t)dB(t) = f(t), \quad x(t_0) = x_0 \tag{2.1}$$

where B(t) is a standard Brownian motion and σ^2 is the intensity of white noise, then

$$x(t) = \frac{x(t_0)}{e^{\int_0^t r(u)du + \sigma B(t) + \frac{1}{2}\sigma^2 t}} + \int_{t_0}^t f(s) e^{\int_0^{s-t} r(u)du + \sigma B(s-t) + \frac{1}{2}\sigma^2 (s-t)} ds.$$

Proof. Setting

$$y(t) = e^{\int_0^t r(u)du + \sigma B(t) + \frac{1}{2}\sigma^2 t}.$$

By lemma 2.1 we have

$$d[x(t)y(t)] = x(t)dy(t) + y(t)dx(t) + dx(t).dy(t). \tag{2.2}$$

Applying Itˆo's formula, we deduce

$$dy(t) = y(t)[r(t) + \sigma^2]dt + \sigma y(t)dB(t) \tag{2.3}$$

According to Eq. (2.1) we have

$$dx(t) = f(t)dt - r(t)x(t)dt - \sigma x(t)dB(t). \tag{2.4}$$

From (2.3) and (2.4), we have

$$dx(t).dy(t) = -\sigma^2 x(t)y(t)dt, \tag{2.5}$$

$$y(t)dx(t) = y(t)f(t)dt - x(t)y(t)[r(t)dt + \sigma dB(t)], \tag{2.6}$$

$$x(t)dy(t) = x(t)y(t)[(r(t) + \sigma^2)dt + \sigma dB(t)]. \tag{2.7}$$

Substituty (2.5), (2.6) and (2.7) into (2.2), leads to

$$d[x(t)y(t)] = x(t)dy(t) + y(t)dx(t) + dx(t).dy(t) = f(t)y(t)dt.$$

Namely

$$d[x(t)e^{\int_0^t r(u)du + \sigma B(t) + \frac{1}{2}\sigma^2 t}] = f(t)e^{\int_0^t r(u)du + \sigma B(t) + \frac{1}{2}\sigma^2 t} dt,$$

Integrating both sides from t_0 to t gives, we get

$$x(t)e^{\int_0^t r(u)du+\sigma B(t)+\frac{1}{2}\sigma^2 t} = x(t_0)+\int_{t_0}^t f(s)e^{\int_0^{s-t} r(u)du+\sigma B(s-t)+\frac{1}{2}\sigma^2 s}ds,$$

which yields

$$x(t)=\frac{x(t_0)}{e^{\int_0^t r(u)du+\sigma B(t)+\frac{1}{2}\sigma^2 t}}+\int_{t_0}^t f(s)e^{\int_0^{s-t} r(u)du+\sigma B(s-t)+\frac{1}{2}\sigma^2 (s-t)}ds.$$

This completes the proof.

Lemma 2.3. ([4]) The B(t) of the Brownian motion has the following properties:

(i) (Continuity) The almost orbit of B(t) is continuous in $[0,+\infty)$

(ii) (Asymptoticity) The Brownian motion of one-dimensional standard satisfy the law of the iterated logarithm, as following:

$$\liminf_{t\to\infty}\frac{B(t)}{\sqrt{2t\ln\ln t}}=-1,\quad a.s.,\qquad \limsup_{t\to\infty}\frac{B(t)}{\sqrt{2t\ln\ln t}}=1,\quad a.s..$$

($C1$) According to Lemma 2.3 (i), for some $T_0>0$, there must exist $K>0$ such that

$$|B(t)|\le K,\forall t\in[0,T_0],a.s..$$

(C2) From Lemma 2.3 (ii), we have $\lim_{t\to\infty}\frac{\mathrm{B(t)}}{\mathrm{t}}=0$. Hence for $\forall\varepsilon>0$, there is $T0>0$ such that $|B(t)|<\varepsilon t,\forall t\in[0,\infty),a.s.,$

In view the ($C1$) and ($C2$), we have the following lemma:

Lemma 2.4. Assume that the B(t) is the Brownian motion, then for $\forall\varepsilon>0$, $|B(t)|\le K+\varepsilon t,\forall t\in[0,\infty),a.s,$ where

$$K=\sup_{s\in[0,T_0]}|B(s)|.$$

According Lemma 2.4, we have

(L_1) for $\forall\varepsilon_1>0$, there must exist $T_1>0$ and $K_1>0$ such That $|B_1(t)|\le K_1+\varepsilon_1 t$ for all $t\ge 0$, where

$$K_1=\sup_{s\in[0,T_1]}|B_1(s)|.$$

(L_2) for $\forall\varepsilon_2>0$, there must exist $T_2>0$ and $K_2>0$ such That $|B_2(t)|\le K_2+\varepsilon_2 t$ for $t\ge 0$, where

$$K_2=\sup_{s\in[0,T_2]}|B_2(s)|.$$

(L_3) for $\forall\varepsilon_3>0$, there must exist $T_3>0$ and $K_3>0$ such That $|B_3(t)|\le K_3+\varepsilon_3 t$ for $t\ge 0$, where

$$K_3=\sup_{s\in[0,T_3]}|B_3(s)|.$$

Here B_1,B_2 and B_3 are defined as that in system (1.2).

3. Almost Sure Permanence

Setting:

$$M_S=\frac{\alpha^+e^{K_1\sigma_1}}{\mu^++\frac{1}{2}\sigma_1^2-\varepsilon_1\sigma_1},\qquad m_S=\frac{k_2\alpha^--\beta^+M_S}{k_2e^{K_1\sigma_1}(\mu^-+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)},$$

$$M_I=\frac{\beta^+M_Se^{K_2\sigma_2}}{k_2(\mu^++\gamma^++\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)},\quad m_I=\frac{\mu^--\sigma}{(2\mu^-+\gamma^-+\frac{1}{2}\sigma_2^2+\varepsilon_2\sigma_2)e^{K_2\sigma_2}},$$

$$M_R=\frac{(\gamma^+M_I+\sigma)e^{K_3\sigma_3}}{\mu^++\frac{1}{2}\sigma_3^2-\varepsilon_3\sigma_3},\qquad m_R=\frac{m_I\gamma^-}{(\mu^-+\frac{1}{2}\sigma_3^2+\varepsilon_3\sigma_3)e^{K_3\sigma_3}}.$$

Lemma 3.1. Assume that (H_1) $k_2\alpha^->\beta^+M_S$. Then the susceptible S(t) of system (1.2) is almost stochastically permanent, that is,

$$0<m_s\le\liminf_{t\to\infty}S(t)\le\limsup_{t\to\infty}S(t)\le M_S<\infty,\qquad a.s.,$$

Where

$$M_S=\frac{\alpha^+e^{K_1\sigma_1}}{\mu^++\frac{1}{2}\sigma_1^2-\varepsilon_1\sigma_1},\quad m_S=\frac{k_2\alpha^--\beta^+M_S}{k_2e^{K_1\sigma_1}(\mu^-+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)}.$$

Proof. From the first equation of system (1.2), we obtain

$$dS(t)=[\alpha(t)-\mu(t)S(t)-\frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)}]dt-\sigma_1S(t)dB_1(t).$$

Thus

$$dS(t)+\mu(t)S(t)+\sigma_1 S(t)dB_1(t)=[\alpha(t)-\frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)}]dt. \tag{3.1}$$

By Lemma 2.2, set $t_0=0$ we obtain

$$S(t)=\frac{S(0)}{e^{\int_0^t \mu(u)du+\sigma_1 B_1(t)+\frac{1}{2}\sigma_1^2 t}}+\int_0^t[\alpha(s)-\frac{\beta(s)S(s)I(s)}{k_1+k_2I(s)}]e^{\int_0^{s-t}\mu(u)du+\sigma_1 B_1(s-t)+\frac{1}{2}\sigma_1^2(s-t)}ds,$$

Which yields

$$S(t)\le\frac{S(0)}{e^{\int_0^t \mu(u)du+\sigma_1 B_1(t)+\frac{1}{2}\sigma_1^2 t}}+\int_0^t\alpha(s)e^{\int_0^{s-t}\mu(u)du+\sigma_1 B_1(s-t)+\frac{1}{2}\sigma_1^2(s-t)}ds\le\frac{S(0)}{e^{\int_0^t \mu(u)du+\sigma_1 B_1(t)+\frac{1}{2}\sigma_1^2 t}}+\alpha^+\int_0^t e^{\int_0^{s-t}\mu(u)du+\sigma_1 B_1(s-t)+\frac{1}{2}\sigma_1^2(s-t)}ds$$

$$\le\frac{S(0)}{e^{(\mu^-+\frac{1}{2}\sigma_1^2)t-\sigma_1|B_1(t)|}}+\alpha^+\int_0^t e^{(\mu^++\frac{1}{2}\sigma_1^2)(s-t)+\sigma_1|B_1(s-t)|}ds.$$

According to (L_1) we have

$$S(t)\le\frac{S(0)}{e^{(\mu^-+\frac{1}{2}\sigma_1^2)t-\sigma_1(K_1+\varepsilon_1 t)}}+\alpha^+\int_0^t e^{(\mu^++\frac{1}{2}\sigma_1^2)(s-t)+\sigma_1[K_1+\varepsilon_1|s-t|]}ds\le\frac{S(0)e^{K_1\sigma_1}}{e^{(\mu^-+\frac{1}{2}\sigma_1^2-\varepsilon_1\sigma_1)t}}+\alpha^+e^{K_1\sigma_1}\int_0^t e^{(\mu^++\frac{1}{2}\sigma_1^2-\varepsilon_1\sigma_1)(s-t)}ds$$

$$\le\frac{S(0)e^{K_1\sigma_1}}{e^{(\mu^-+\frac{1}{2}\sigma_1^2-\varepsilon_1\sigma_1)t}}+\frac{\alpha^+e^{K_1\sigma_1}}{\mu^++\frac{1}{2}\sigma_1^2-\varepsilon_1\sigma_1}[1-\frac{1}{e^{(\mu^++\frac{1}{2}\sigma_1^2-\varepsilon_1\sigma_1)t}}].$$

We note that $(\mu^++\frac{1}{2}\sigma_1^2-\varepsilon_1\sigma_1)>0$, letting $t\to\infty$ yields that

$$\limsup_{t\to\infty}S(t)\le\frac{\alpha^+e^{K_1\sigma_1}}{\mu^++\frac{1}{2}\sigma_1^2-\varepsilon_1\sigma_1}:=M_S<\infty,\qquad a.s.. \tag{3.2}$$

On the other hand, according to (3.2), for $\forall\varepsilon_1^{'}>0$, there exists $T_1^{'}>0$ such that

$$S(t)\le M_S+\varepsilon_1^{'}\ \text{ for }\ t\ge T_1^{'}.$$

According to Eq. (3.1) we have

$$dS(t)+\mu(t)S(t)+\sigma_1 S(t)dB_1(t)=[\alpha(t)-\frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)}]dt.$$

By Lemme 2.2, set $t_0=T_1^{'}$ we have

$$S(t)=\frac{S(T_1^{'})}{e^{\int_0^t \mu(u)du+\sigma_1 B_1(t)+\frac{1}{2}\sigma_1^2 t}}+\int_{T_1^{'}}^t[\alpha(s)-\frac{\beta(s)S(s)I(s)}{k_1+k_2I(s)}]e^{\int_0^{s-t}\mu(u)du+\sigma_1 B_1(s-t)+\frac{1}{2}\sigma_1^2(s-t)}ds.$$

Then

$$S(t)\ge\frac{S(T_1^{'})}{e^{\int_0^t \mu(u)du+\sigma_1 B_1(t)+\frac{1}{2}\sigma_1^2 t}}+[\alpha^--\frac{\beta^+(M_S+\varepsilon_1^{'})}{k_2}]\int_{T_1^{'}}^t e^{\int_0^{s-t}\mu(u)du+\sigma_1 B_1(s-t)+\frac{1}{2}\sigma_1^2(s-t)}ds$$

$$\ge\frac{S(T_1^{'})}{e^{(\mu^++\frac{1}{2}\sigma_1^2)t+\sigma_1|B_1(t)|}}+[\alpha^--\frac{\beta^+(M_S+\varepsilon_1^{'})}{k_2}]\int_{T_1^{'}}^t e^{(\mu^-+\frac{1}{2}\sigma_1^2)(s-t)-\sigma_1|B_1(s-t)|}ds.$$

According to (L_1), we have

$$S(t) \geq \frac{S(T_1^{'})}{e^{(\mu^{+}+\frac{1}{2}\sigma_1^2)t+\sigma_1(K_1+\varepsilon_1 t)}} + [\alpha^{-} - \frac{\beta^{+}(M_S+\varepsilon_1^{'})}{k_2}]\int_{T_1^{'}}^{t} e^{(\mu^{-}+\frac{1}{2}\sigma_1^2)(s-t)-\sigma_1[K_1+\varepsilon_1|s-t|]} ds$$

$$\geq \frac{S(T_1^{'})}{e^{(\mu^{+}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)t+K_1\sigma_1}} + [\alpha^{-} - \frac{\beta^{+}(M_S+\varepsilon_1^{'})}{k_2}]\int_{T_1^{'}}^{t} e^{(\mu^{-}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)(s-t)-K_1\sigma_1} ds \geq \frac{S(T_1^{'})e^{-K_1\sigma_1}}{e^{(\mu^{+}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)t}} + \frac{k_2\alpha^{-}-\beta^{+}(M_S+\varepsilon_1^{'})}{k_2 e^{K_1\sigma_1}}\int_{T_1^{'}}^{t} e^{(\mu^{-}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)(s-t)} ds$$

$$\geq \frac{S(T_1^{'})e^{-K_1\sigma_1}}{e^{(\mu^{+}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)t}} + \frac{k_2\alpha^{-}-\beta^{+}(M_S+\varepsilon_1^{'})}{k_2 e^{K_1\sigma_1}(\mu^{-}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)}[1-\frac{1}{e^{(\mu^{-}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)(t-T_1^{'})}}].$$

Letting $\varepsilon_1^{'} \to 0$ we have

$$S(t) \geq \frac{S(T_1^{'})e^{-K_1\sigma_1}}{e^{(\mu^{+}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)t}} + \frac{k_2\alpha^{-}-\beta^{+}M_S}{k_2 e^{K_1\sigma_1}(\mu^{-}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)}[1-\frac{1}{e^{(\mu^{-}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)(t-T_1^{'})}}].$$

We note that $k_2\alpha^{-} > \beta^{+}M_S$, letting $t \to \infty$, we get

$$\liminf_{t\to\infty} S(t) \geq \frac{k_2\alpha^{-}-\beta^{+}M_S}{k_2 e^{K_1\sigma_1}(\mu^{-}+\frac{1}{2}\sigma_1^2+\varepsilon_1\sigma_1)} > 0, \quad a.s.. \tag{3.3}$$

(3.2) and (3.3) yields the required assertion. This completes the proof.

Lemma 3.2. Assume that (H_2) $\mu^{-} > \sigma$. Then the infective I(t) of system (1.2), is almost stochastically permanent, that is,

$$0 < m_I \leq \liminf_{t\to\infty} I(t) \leq \limsup_{t\to\infty} I(t) \leq M_I < \infty, \quad a.s.,$$

Where

$$M_I = \frac{\beta^{+}M_S e^{K_2\sigma_2}}{k_2(\mu^{+}+\gamma^{+}+\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)}, \quad m_I = \frac{\mu^{-}-\sigma}{(2\mu^{-}+\gamma^{-}+\frac{1}{2}\sigma_2^2+\varepsilon_2\sigma_2)e^{K_2\sigma_2}}.$$

Proof. From the second equation of system (1.2), we obtain

$$dI(t) = [\frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)} - (\mu(t)+\gamma(t))I(t) - h(I)]dt - \sigma_2 I(t)dB_2(t). \tag{3.4}$$

Thus

$$dI(t) + (\mu(t)+\gamma(t))I(t)dt + \sigma_2 I(t)dB_2(t) = [\frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)} - h(I)]dt.$$

By Lemme 2.2, set $t_0 = T_1^{'}$ we have

$$I(t) = \frac{I(T_1^{'})}{e^{\int_0^t(\mu(u)+\gamma(u))du+\sigma_2B_2(t)+\frac{1}{2}\sigma_2^2 t}} + \int_{T_1^{'}}^{t}[\frac{\beta(s)S(s)I(s)}{k_1+k_2I(s)} - h(I)]e^{\int_0^{s-t}(\mu(u)+\gamma(u))du+\sigma_2B_2(s-t)+\frac{1}{2}\sigma_2^2(s-t)} ds$$

$$\leq \frac{I(T_1^{'})}{e^{\int_0^t(\mu(u)+\gamma(u))du+\sigma_2B_2(t)+\frac{1}{2}\sigma_2^2 t}} + \frac{\beta^{+}(M_S+\varepsilon_1^{'})}{k_2}\int_{T_1^{'}}^{t} e^{\int_0^{s-t}(\mu(u)+\gamma(u))du+\sigma_2B_2(s-t)+\frac{1}{2}\sigma_2^2(s-t)} ds$$

$$\leq \frac{I(T_1^{'})}{e^{(\mu^{-}+\gamma^{-}+\frac{1}{2}\sigma_2^2)t-\sigma_2|B_2(t)|}} + \frac{\beta^{+}(M_S+\varepsilon_1^{'})}{k_2}\int_{T_1^{'}}^{t} e^{(\mu^{+}+\gamma^{+}+\frac{1}{2}\sigma_2^2)(s-t)-\sigma_2|B_2(s-t)|} ds.$$

According to (L_2) we get

$$I(t)\le\frac{I(T_1^{'})}{e^{(\mu^-+\gamma^-+\frac{1}{2}\sigma_2^2)t-\sigma_2(K_2+\varepsilon_2 t)}}+\frac{\beta^+(M_S+\varepsilon_1^{'})}{k_2}\int_{T_1^{'}}^{t}e^{(\mu^++\gamma^++\frac{1}{2}\sigma_2^2)(s-t)-\sigma_2(K_2+\varepsilon_2|s-t|)}ds$$

$$\le\frac{I(T_1^{'})e^{K_2\sigma_2}}{e^{(\mu^-+\gamma^-+\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)t}}+\frac{\beta^+(M_S+\varepsilon_1^{'})e^{K_2\sigma_2}}{k_2}\int_{T_1^{'}}^{t}e^{(\mu^++\gamma^++\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)(s-t)}ds$$

$$\le\frac{I(T_1^{'})e^{K_2\sigma_2}}{e^{(\mu^-+\gamma^-+\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)t}}+\frac{\beta^+(M_S+\varepsilon_1^{'})e^{K_2\sigma_2}}{k_2(\mu^++\gamma^++\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)}[1-\frac{1}{e^{(\mu^++\gamma^++\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)(t-T_1^{'})}}].$$

Let $\varepsilon_1^{'}\to 0$, we get

$$I(t)\le\frac{I(T_1^{'})e^{K_2\sigma_2}}{e^{(\mu^-+\gamma^-+\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)t}}+\frac{\beta^+M_Se^{K_2\sigma_2}}{k_2(\mu^++\gamma^++\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)}[1-\frac{1}{e^{(\mu^++\gamma^++\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)(t-T_1^{'})}}].$$

Taking $\varepsilon_2>0$, to satisfy $(\mu^++\gamma^++\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)>0$, letting $t\to\infty$, yields that

$$\limsup_{t\to\infty}I(t)\le\frac{\beta^+M_Se^{K_2\sigma_2}}{k_2(\mu^++\gamma^++\frac{1}{2}\sigma_2^2-\varepsilon_2\sigma_2)}:=M_I<\infty\qquad a.s. \tag{3.5}$$

On the other hand, according to (3.4) we have

$$dI(t)=[\frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)}-(\mu(t)+\gamma(t))I(t)-h(I)]dt-\sigma_2I(t)dB_2(t)$$

$$=[\frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)}+\mu(t)-(2\mu(t)+\gamma(t))I(t)-h(I)]dt-\sigma_2I(t)dB_2(t)$$

Thus

$$dI(t)+(2\mu(t)+\gamma(t))I(t)+\sigma_2I(t)dB_2(t)=[\frac{\beta(t)S(t)I(t)}{k_1+k_2I(t)}+\mu(t)-h(I)]dt.$$

By Lemma 2.2, let $t_0=0$ we obtain

$$I(t)=\frac{I(0)}{e^{\int_0^t(2\mu(u)+\gamma(u))du+\sigma_2B_2(t)+\frac{1}{2}\sigma_2^2t}}+\int_0^t[\frac{\beta(s)S(s)I(s)}{k_1+k_2I(s)}+\mu(s)-h(I)]e^{\int_0^{s-t}(2\mu(u)+\gamma(u))du+\sigma_2B_2(s-t)+\frac{1}{2}\sigma_2^2(s-t)}ds$$

$$\ge\frac{I(0)}{e^{\int_0^t(2\mu(u)+\gamma(u))du+\sigma_2|B_2(t)|+\frac{1}{2}\sigma_2^2t}}+\int_0^t(\mu(s)-h(I))e^{\int_0^{s-t}(2\mu(u)+\gamma(u))du+\sigma_2|B_2(s-t)|+\frac{1}{2}\sigma_2^2(s-t)}ds$$

$$\ge\frac{I(0)}{e^{\int_0^t(2\mu(u)+\gamma(u))du+\sigma_2|B_2(t)|+\frac{1}{2}\sigma_2^2t}}+(\mu^--\sigma)\int_0^te^{\int_0^{s-t}(2\mu(u)+\gamma(u))du+\sigma_2|B_2(s-t)|+\frac{1}{2}\sigma_2^2(s-t)}ds.$$

According to (L_2), we have

$$I(t)\ge\frac{I(0)}{e^{(2\mu^++\gamma^++\frac{1}{2}\sigma_2^2)t+\sigma_2(K_2+\varepsilon_2t)}}+(\mu^--\sigma)\int_0^te^{(2\mu^-+\gamma^-+\frac{1}{2}\sigma_2^2)(s-t)-\sigma_2(K_2+\varepsilon_2|s-t|)}ds\ge\frac{I(0)e^{-K_2\sigma_2}}{e^{(2\mu^++\gamma^++\frac{1}{2}\sigma_2^2+\varepsilon_2\sigma_2)t}}+\frac{\mu^--\sigma}{e^{K_2\sigma_2}}\int_0^te^{(2\mu^-+\gamma^-+\frac{1}{2}\sigma_2^2+\varepsilon_2\sigma_2)(s-t)}ds$$

$$\ge\frac{I(0)e^{-K_2\sigma_2}}{e^{(2\mu^++\gamma^++\frac{1}{2}\sigma_2^2+\varepsilon_2\sigma_2)t}}+\frac{\mu^--\sigma}{(2\mu^-+\gamma^-+\frac{1}{2}\sigma_2^2+\varepsilon_2\sigma_2)e^{K_2\sigma_2}}[1-\frac{1}{e^{(2\mu^-+\gamma^-+\frac{1}{2}\sigma_2^2+\varepsilon_2\sigma_2)t}}].$$

We note that $\mu^- > \sigma$, letting $t \to \infty$, yields that

$$\liminf_{t\to\infty} I(t) \geq \frac{\mu^- - \sigma}{(2\mu^- + \gamma^- + \frac{1}{2}\sigma_2^2 + \varepsilon_2\sigma_2)e^{K_2\sigma_2}} := m_I > 0 \qquad a.s.. \tag{3.6}$$

(3.5) and (3.6) yields the required assertion. This completes the proof.

Lemma 3.3. The remove R(t) of system (1.2), is almost stochastically permanent, that is,

$$0 < m_R \leq \liminf_{t\to\infty} R(t) \leq \limsup_{t\to\infty} R(t) \leq M_R < \infty, \qquad a.s.,$$

Where

$$M_R = \frac{(\gamma^+ M_I + \sigma)e^{K_3\sigma_3}}{\mu^+ + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3}, \quad m_R = \frac{m_I\gamma^-}{(\mu^- + \frac{1}{2}\sigma_3^2 + \varepsilon_3\sigma_3)e^{K_3\sigma_3}}.$$

Proof. According to (3.5), for $\forall \varepsilon_2^{'} > 0$, there exists $T_2^{'} > 0$ such that

$$I(t) \leq M_I + \varepsilon_2^{'} \quad \text{for} \quad t \geq T_2^{'}$$

From the third equation of system (1.2), we have

$$dR(t) = [\gamma(t)I(t) + h(I) - \mu(t)R(t)]dt - \sigma_3 R(t)dB_3(t).$$

Then

$$dR(t) + \mu(t)R(t) + \sigma_3 R(t)dB_3(t) = [\gamma(t)I(t) + h(I)]dt. \tag{3.7}$$

By Lemma 2.2, set $t_0 = T_2^{'}$ we have

$$\begin{aligned} R(t) &= \frac{R(T_2^{'})}{e^{\int_0^t \mu(u)du + \sigma_3 B_3(t) + \frac{1}{2}\sigma_3^2 t}} + \int_{T_2^{'}}^t (\gamma(s)I(s) + h(I))e^{\int_0^{s-t}\mu(u)du + \sigma_3 B_3(s-t) + \frac{1}{2}\sigma_3^2(s-t)}ds \\ &\leq \frac{R(T_2^{'})}{e^{(\mu^- + \frac{1}{2}\sigma_3^2)t - \sigma_3|B_3(t)|}} + [\gamma^+(M_I + \varepsilon_2^{'}) + \sigma]\int_{T_2^{'}}^t e^{(\mu^+ + \frac{1}{2}\sigma_3^2)(s-t) + \sigma_3|B_3(s-t)|}ds. \end{aligned}$$

According to (L_3) we get

$$R(t) \leq \frac{R(T_2^{'})}{e^{(\mu^- + \frac{1}{2}\sigma_3^2)t - \sigma_3(K_3 + \varepsilon_3 t)}} + [\gamma^+(M_I + \varepsilon_2^{'}) + \sigma]\int_{T_2^{'}}^t e^{(\mu^+ + \frac{1}{2}\sigma_3^2)(s-t) + \sigma_3(K_3 + \varepsilon_3|s-t|)}ds$$

$$\leq \frac{R(T_2^{'})e^{K_3\sigma_3}}{e^{(\mu^- + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3)t}} + [\gamma^+(M_I + \varepsilon_2^{'}) + \sigma]e^{K_3\sigma_3}\int_{T_2^{'}}^t e^{(\mu^+ + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3)(s-t)}ds \leq \frac{R(T_2^{'})e^{K_3\sigma_3}}{e^{(\mu^- + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3)t}} + \frac{[\gamma^+(M_I + \varepsilon_2^{'}) + \sigma]e^{K_3\sigma_3}}{\mu^+ + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3}[1 - \frac{1}{e^{(\mu^+ + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3)(t - T_2^{'})}}].$$

Letting $\varepsilon_2^{'} \to 0$, we have

$$R(t) \leq \frac{R(T_2^{'})e^{K_3\sigma_3}}{e^{(\mu^- + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3)t}} + \frac{(\gamma^+ M_I + \sigma)e^{K_3\sigma_3}}{\mu^+ + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3}[1 - \frac{1}{e^{(\mu^+ + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3)(t - T_2^{'})}}].$$

Taking $\varepsilon_3 > 0$ to satisfy $(\mu^- + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3) > 0$, letting $t \to \infty$, then we have

$$\limsup_{t\to\infty} R(t) \leq \frac{(\gamma^+ M_I + \sigma)e^{K_3\sigma_3}}{\mu^+ + \frac{1}{2}\sigma_3^2 - \varepsilon_3\sigma_3} := M_R < \infty \qquad a.s.. \tag{3.8}$$

On the other hand, according to (3.6), for $\forall \varepsilon_3^{'} > 0$ there exists $T_3^{'} > 0$ such that

$$I(t) \geq m_I - \varepsilon_3^{'} \quad \text{for} \quad t \geq T_3^{'}$$

According to Eq. (3.7), we have

$$dR(t) + \mu(t)R(t) + \sigma_3 R(t)dB_3(t) = [\gamma(t)I(t) + h(I)]dt.$$

By lemma 2.2 set $t_0 = T_3^{'}$, we have

$$R(t) = \frac{R(T_3^{'})}{e^{\int_0^t \mu(u)du + \sigma_3 B_3(t) + \frac{1}{2}\sigma_3^2 t}} + \int_{T_3^{'}}^t [\gamma(s)I(s) + h(I)]e^{\int_0^{s-t} \mu(u)du + \sigma_3 B_3(s-t) + \frac{1}{2}\sigma_3^2 (s-t)} ds$$
$$\geq \frac{R(T_3^{'})}{e^{(\mu^+ + \frac{1}{2}\sigma_3^2)t + \sigma_3 |B_3(t)|}} + (m_I - \varepsilon_3^{'})\gamma^- \int_{T_3^{'}}^t e^{(\mu^- + \frac{1}{2}\sigma_3^2)(s-t) - \sigma_3 |B_3(s-t)|} ds.$$

According to (L_3), we have

$$R(t) \geq \frac{R(T_3^{'})}{e^{(\mu^+ + \frac{1}{2}\sigma_3^2)t + \sigma_3 (K_3 + \varepsilon_3 t)}} + (m_I - \varepsilon_3^{'})\gamma^- \int_{T_3^{'}}^t e^{(\mu^- + \frac{1}{2}\sigma_3^2)(s-t) - \sigma_3 (K_3 + \varepsilon_3 |s-t|)} ds \geq \frac{R(T_3^{'})e^{-K_3\sigma_3}}{e^{(\mu^+ + \frac{1}{2}\sigma_3^2 + \varepsilon_3\sigma_3)t}} + \frac{(m_I - \varepsilon_3^{'})\gamma^-}{e^{K_3\sigma_3}} \int_{T_3^{'}}^t e^{(\mu^- + \frac{1}{2}\sigma_3^2 + \varepsilon_3\sigma_3)(s-t)} ds$$
$$\geq \frac{R(T_3^{'})e^{-K_3\sigma_3}}{e^{(\mu^+ + \frac{1}{2}\sigma_3^2 + \varepsilon_3\sigma_3)t}} + \frac{(m_I - \varepsilon_3^{'})\gamma^-}{(\mu^- + \frac{1}{2}\sigma_3^2 + \varepsilon_3\sigma_3)e^{K_3\sigma_3}}[1 - \frac{1}{e^{(\mu^- + \frac{1}{2}\sigma_3^2 + \varepsilon_3\sigma_3)(t - T_3^{'})}}].$$

Letting $\varepsilon_3^{'} \to 0$, we have

$$R(t) \geq \frac{R(T_3^{'})e^{-K_3\sigma_3}}{e^{(\mu^+ + \frac{1}{2}\sigma_3^2 + \varepsilon_3\sigma_3)t}} + \frac{m_I\gamma^-}{(\mu^- + \frac{1}{2}\sigma_3^2 + \varepsilon_3\sigma_3)e^{K_3\sigma_3}}[1 - \frac{1}{e^{(\mu^- + \frac{1}{2}\sigma_3^2 + \varepsilon_3\sigma_3)(t - T_3^{'})}}].$$

Letting $t \to \infty$, then we have

$$\liminf_{t\to\infty} R(t) \geq \frac{m_I\gamma^-}{(\mu^- + \frac{1}{2}\sigma_3^2 + \varepsilon_3\sigma_3)e^{K_3\sigma_3}} := m_R > 0 \quad a.s.. \tag{3.9}$$

(3.8) and (3.9) yields the required assertion. This completes the proof.

According to Lemma 3.1, Lemma 3.2 and Lemma 3.3, we have the following theorem:

Theorem 3.1. If (H1) and (H2) are hold, then system (1.2) is almost surely stochastically permanent.

4. Conclusion

This paper is concerned with the permanence of the non-autonomous stochastic SIR model. The results in this paper imply that the intensity of white noise (i.e., σ_i^2 , i=2,3) has no effect on the permanence of the infective and the removed class of system (1.2). By Lemma 3.1, it is easy to see that the intensity of white noise (i.e., σ_1^2) on the susceptible has some influence On the permanence of the susceptible of system (1.2). From Lemma 3.2, we observe that if the natural death rate μ exceeds the positive constant σ, the infective is permanent. Without any condition, the removed class of system (1.2) is always persistent.

References

[1] Z.G. Bai and Y.C. Zhou, Existence of two perriodic solutions for a non-autonomous SIR epidemic model, Appl. Math. Model. 35 (2011) 382-391.

[2] O. Diekmann, J.A.P. Heesterbeek and J.A.J. Metz, On the definition and computation of the basic reproductive ratio in models for infectious in heterogeneous population, J. Math. Biol. 28 (1990) 365-382.

[3] H.W. Hethcote, The mathematics of infectious diseases, Siam Review 42 (2000) 599-653.

[4] S.G. Hu, C.M. Huang and F.K. Wu, Stochastic Differential Equations, Science Publishing House, Beijing, 2008 (Chinese).

[5] W.O. Kermack and A.G. McKendrick, Contribution to the mathematical theory of epidemics, Proc. Roy. Soc. A 115 (1927) 700-721.

[6] S.Y. Liu, Y.Z. Pei, C.G. Li and L.S. Chen, Three kinds of TVS in a SIR epidemic model with saturated infectious force and vertical transmission, Appl. Math. Model. 33 (2009) 1923-1932.

[7] J. Lv, K. Wang, Definition on almost sure permanence of stochastic single species models, J. Math. Anal. Appl. 422 (2015) 675-683.

[8] R.M. May, Stability and Complexity in Model Ecosystems, Princeton University Press, NJ, 2001.

[9] Y.Z. Pei, S.P. Li, C.G. Li and S.Z. Chen, The effect of constant and pulse vaccination on an SIR epidemic model with infectious period, Appl. Math. Model. 35 (2011) 3866-3878.

[10] L. Stone, B. Shulgin and Z. Agur, Theoretical examination of the pulse vaccination policy in the SIR epidemic model, Math. Comput. Model. 31 (2000) 207-215.

[11] R.W. West and J.R. Thompson, Models for the simple epidemic, Math. Biosci. 141 (1997) 29-39.

[12] R. Xu and Z.E. Ma, Global stability of a SIR epidemic model with nonlinear incidencerate and time delay, Nonlinear Anal.: RWA 10 (2009) 3175-3189. 10.