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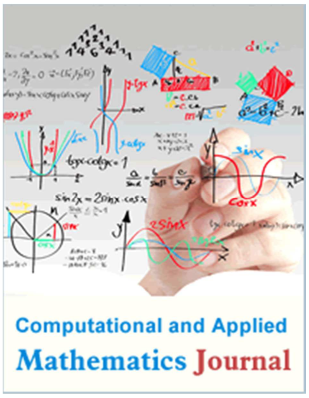

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Partial Chain Topologies on Finite Sets

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Abstract
Although, there are lots of equivalent ways of formulating for computing the number of topological spaces in finite set. In this paper, we formulated special case for computing the number of chain topological spaces, and maximal elements with the natural generalization. We look at the concept of partial chain topologies on finite set with respect to the given subset. We determine the number of partial chain topologies with *k* open sets, and so the number of all chain topologies on finite set will be outlined. To support our study, some examples, and properties of this concept will be studied. Moreover, we determine the rule for computing the number of all maximal elements in the set of all chain topologies.

1. Introduction

A topology τ on the set $X \neq \phi$ is a subset of the power set $P(X)$ that contains ϕ and X, and is closed under arbitrary unions and finite intersections. The number of topologies on a finite set is a problem that has been worked on by many Mathematicians. Moussa Benoumhani in [9], computed the number of topologies having k open sets $T(n,k)$ on the finite set X, having n elements for $2 \leq k \leq 12$. Ern´e and Stege [4] provided the best methods, and gave the number of topologies on an n element set up to $n = 14$. B. Richmond [10] discussed method for counting finite topologies is to count associated quasiorders, and he proved that every quasiorder gives a topology and in the other direction, every topology gives a quasiorder. T. Richmond [11] discussed the idea of principal topologies or Alexandroff topologies. A *principal topology* is a topology that is closed under arbitrary intersections. Since we are working on a finite set, every topology on a finite set must be a principal topology because the arbitrary intersections will be finite which must be open from the definition of a topology. A chain topology on X, is a topology whose open sets are totally ordered by inclusion. Stephen in [17], proved that the number of chain topologies on X, having k open sets take the form

$$C(n,k) = \sum_{r=1}^{n-1} \binom{n}{r} C(r,k-1) = (k-1)!\, S(n,k-1)$$

Where

$$S(n,k) = \frac{1}{k!} \sum_{t=0}^{k} (-1)^t \binom{k}{t} (k-r)^n$$

In this paper, we look at the concept of partial chain topologies on any finite set with respect to the given subset. In the second section of this paper, we study the concept of partial chain topology on any finite set with respect to the given subset, and study some properties with respect to some concepts of topological spaces. In the third section of this paper, we find the number of all partial chain topologies with respect to the given subset A to help us in obtaining the number of all chain topologies on it. Moreover, many examples will be studied. In the fourth section of this paper, we construct all maximal elements in the set of all chain topologies on finite set, and so we find the number of them.

2. Partial Chain Topology with Respect to Sets

In this section, we study the concept of partial chain topology on finite set with respect to the given subset, and study some properties with respect to some concepts of topological spaces.

Definition 2.1. Let X be a finite set having n elements, for every $A \subset X$, the partial chain topology on X with respect to A is the chain topology on X which is defined as:

$$\tau_A = \{X, \phi, U : A \subset U\}$$

We can rewrite this concept as:

$$\tau_A = \{X, \phi, A = U_0, U_1, U_2, U_3, \ldots, U_k \,:\, A \subset U_1 \subset U_2 \subset U_3 \subset \cdots \subset U_k\}$$

It is clear that τ_X, having only indiscrete partial chain topology $\{X, \phi\}$, and τ_ϕ, having any chain topology on the set X as a partial chain topology with respect to it. Therefore, in our study, we avoid the set $A \subset X$ to be X or ϕ.

It is interested to show that the intersection of arbitrary partial chain topologies with respect to A is the partial chain topology with respect to it. Moreover, the union of partial chain topologies with respect to A is not in general a partial chain topology with respect to it, as studying in the following example.

Example 2.1. Let the set $X = \{x_1, x_2, x_3\}$, and A be the subset of X, where $A = \{x_1\}$. Then $\tau_{\{x_1\}} = \{X, \phi, \{x_1\}, \{x_1, x_2\}\}$ and $u_{\{x_1\}} = \{X, \phi, \{x_1\}, \{x_1, x_3\}\}$ are two partial chain topologies with respect to A, and

$$\tau_{\{x_1\}} \cup u_{\{x_1\}} = \{X, \phi, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}\}$$

is a topology on the set X, and not a partial chain topology with respect to A.

Theorem 2.1. The family of closed sets of any partial chain topology with respect to A is a partial chain topology with respect to B, where B is the complement of maximum of all improper open subsets of X with respect to the inclusion relation.

Theorem 2.2. Let X be a finite set having n elements, and let τ_A be the partial chain topology on X with respect to A. Then any subset $V \in \tau_A$ is a dense subset.

Proof. The proof is easy, since, if the closure of set V is $\bar{V} = U$, and $U \neq X$, then from Theorem (2.1), $V \subset U \subset A^C$. This contradicts $A \subset V$, and so the set U must be X. Therefore $V \in \tau_A$ is a dense subset.

It is clear that the partial chain topological space with respect to any subset A of the nonempty set X is compact, Moreover, any partial chain topological space with respect to any subset A of X is connected space, since the only clopen (open and closed) subsets of the space X are X, ϕ.

Theorem 2.3. The relative topology on a subspace B of a partial chain topological space with respect to the subset A on the set X is a partial chain topological space with respect to the subset $A \cap B$ on the set B.

Theorem 2.4. Homeomorphic image of a partial chain topological space with respect to the subset A on the set X is a partial chain topological space with respect to the image of A on the image of X.

3. Number of Partial Chain Topology with Respect to Sets

In this section we find the number of all partial chain topologies with respect to the given subset A to help us in obtaining the number of all chain topologies on it. Moreover, many examples will be studied.

Notations: Let X be a finite set having n elements. In our study, the set of all partial chain topology with respect to $A \subset X$ with k number of open sets will be denoted by $c_A(n,k)$, $c_r(n,k)$ is the set of all partial chain topologies of all subsets of X, having r elements with k open sets, Moreover, the number of all partial chain topology with respect to $A \subset X$ with k number of open sets will be denoted by $C_A(n,k) = |c_A(n,k)|$, $C_r(n,k) = |c_r(n,k)|$ is the number of all partial chain topologies of all subsets of X, having r elements with k open sets, Also, $C_A(n)$ is the number of all partial chain topology with respect to $A \subset X$, $C_r(n)$ is the number of all partial chain topologies of all subsets of X, having r elements, and $C(n)$ is the number of all chain topologies on X.

It is clear that the number k is a positive integer number, where $3 \leq k \leq n+1$. Moreover, $P(n,r)$ is the notation of permutations, and $\binom{n}{r}$ is the notation of combinations.

Theorem 3.1. Let X be the set having n elements, and let $A \subset X$, where $|A|$ is the number of elements of A. Then the number of all partial chain topologies with respect to A with k open sets is defined as:

$$C_A(n,k) = P(n-|A|, k-3), \text{ where } n-|A|+2 \geq k \geq 3, n-1 \geq |A| \geq 1$$

Proof. Let $\tau_A \in c_A(n,k)$, then $\tau_A = \{X, \varphi, A, U_1, U_2, U_3, \ldots, U_{(k-3)}\}$. We can choose U_1 with $(n-|A|)$ methods, and U_2 with $(n-|A|-1)$ methods, and U_s with $(n-|A|+1-s)$ methods, where $k-3 \geq s \geq 3$. Therefore, by using the principle counting rule, we can choose $c_A(n,k)$ with number of methods equal $(n-|A|)(n-|A|-1)(n-|A|-2)\ldots(n-|A|+1-(k-3))$, which is the permutation $P(n-|A|, k-3)$. Which implies that

$$C_A(n,k) = |c_A(n,k)| = P(n-|A|, k-3)$$

Corollary 3.1. The number of all partial chain topologies with respect to A is

$$C_A(n) = \sum_{k=3}^{n-|A|+2} P(n-|A|, k-3)$$

Theorem 3.2. The number of all partial chain topologies with respect to all subsets of *X*, having r elements with *k* open sets is defined as:

$$C_r(n,k) = \frac{P(n, k+r-3)}{r!}, \text{ where } n-r+2 \geq k \geq 3,\ n-1 \geq r \geq 1$$

Proof. From Theorem (3.1)

$$C_r(n,k) = \sum_{|A|=r} C_A(n,k) = \sum_{|A|=r} P(n-|A|, k-3)$$

Since the number of all subsets of the set *X*, having *r* elements is the combination $\binom{n}{r}$, then by using the principle counting rule. It is follows that

$$C_r(n,k) = \binom{n}{r} P(n-r, k-3) = \frac{P(n,r)P(n-r,k-3)}{r!}$$

Since $P(n,r)P(n-r,k-3) = P(n,k+r-3)$, then

$$C_r(n,k) = \frac{P(n,k+r-3)}{r!}, \text{ where } n-r+2 \geq k \geq 3, n-1 \geq r \geq 1$$

Corollary 3.2. The number of all partial chain topologies with respect to all subsets of *X*, having *r* elements is defined as:

$$C_r(n) = \frac{1}{r!} \sum_{k=3}^{n-r+2} P(n, k+r-3)$$

Corollary 3.3. The number of all chain topologies on the finite set *X*, having n elements is defined as:

$$C(n) = \sum_{r=1}^{n-1} C_r(n) = \sum_{r=1}^{n-1} \left(\frac{1}{r!} \sum_{k=3}^{n-r+2} P(n, k+r-3) \right)$$

In the following two theorems we find the general formula for computing $C_r(n,k), C_r(n)$ by using recursively step.

Theorem 3.3.

$$C_{r+1}(n,k) = \frac{(n-k-r+3)C_r(n,k)}{r+1}, \text{ where } n-2 \geq r \geq 1,$$

$$C_1(n,k) = P(n,k-2), n+1 \geq k \geq 3$$

Proof. From Theorem (3.2), we have that

$$C_r(n,k) = \frac{P(n,k+r-3)}{r!}$$

Since

$$P(n,k+r-3) = n(n-1)(n-2) \ldots \ldots (n-k-r+5)(n-k-r+4)$$

$$= \frac{n(n-1)(n-2) \ldots \ldots (n-k-r+5)(n-k-r+4)(n-k-r+3)}{(n-k-r+3)} = \frac{P(n,k+r-2)}{(n-k-r+3)}$$

and $P(n,k+r-2) = (r+1)!\ C_{r+1}(n,k)$, then

$$C_r(n,k) = \frac{(r+1)!\ C_{r+1}(n,k)}{r!\,(n-k-r+3)} = \frac{(r+1)\ C_{r+1}(n,k)}{(n-k-r+3)}$$

Therefore,

$$C_{r+1}(n,k) = \frac{(n-k-r+3)C_r(n,k)}{r+1}, \text{ where } n-2 \geq r \geq 1,$$

$$C_1(n,k) = P(n,k-2), n+1 \geq k \geq 3$$

Theorem 3.4. $C_{r+1}(n) = \frac{C_r(n)-\binom{n}{r}}{r+1}$, where $n-2 \geq r \geq 1$,

$$C_1(n) = P(n,1) + P(n,2) + P(n,3) + \cdots + P(n,n-1)$$

Proof. From Corollary (3.2), we have that

$$C_r(n) = \frac{1}{r!} \sum_{k=3}^{n-r+2} P(n,k+r-3) = \frac{1}{r!} \left(P(n,r) + \sum_{k=4}^{n-r+2} P(n,k+r-3) \right)$$

$$= \frac{1}{r!} \left(P(n,r) + \sum_{k=3}^{n-r+1} P(n,k+r-2) \right) = \frac{1}{r!} \left(P(n,r) + (r+1)!\ C_{r+1}(n) \right)$$

$$= \frac{P(n,r)}{r!} + \frac{(r+1)!\ C_{r+1}(n)}{r!} = \binom{n}{r} + (r+1)\ C_{r+1}(n)$$

Therefore,

$$C_{r+1}(n) = \frac{C_r(n) - \binom{n}{r}}{r+1}, \text{ where } n-2 \geq r \geq 1,$$

$$C_1(n) = P(n,1) + P(n,2) + P(n,3) + \cdots + P(n,n-1)$$

Example 3.1. *We tabulate the values of* $C_1(n,k), C_1(n)$ *for* $2 \leq n \leq 10, 3 \leq k \leq 11$.

n \ k	3	4	5	6	7	8	9	10	11	$C_1(n)$
2	2									2
3	3	6								9
4	4	12	24							40
5	5	20	60	120						205
6	6	30	120	360	720					1236
7	7	42	240	840	2520	5040				8689
8	8	56	336	1680	6720	20160	40320			69280
9	9	72	504	3024	15120	60480	181440	362880		623529
10	10	90	720	5040	30240	151200	604800	1814400	3628800	6235300

Example 3.2. *We tabulate the values of* $C_2(n,k), C_2(n)$ *for* $3 \leq n \leq 10, 3 \leq k \leq 10$.

n \ k	3	4	5	6	7	8	9	10	$C_2(n)$
3	3								3
4	6	12							18
5	10	30	60						100
6	15	60	180	360					615
7	21	105	420	1260	2520				4326
8	28	168	840	3360	10080	20160			34636
9	36	252	1512	7560	30240	90720	181440		311760
10	45	360	2880	20160	120960	483840	1451520	2903040	4982805

Example 3.3. *We tabulate the values of* $C_3(n,k), C_3(n)$ *for* $4 \leq n \leq 10, 3 \leq k \leq 9$.

n \ k	3	4	5	6	7	8	9	$C_3(n)$
4	4							4
5	10	20						30
6	20	60	120					200
7	35	140	420	840				1435
8	56	280	1120	3360	6720			11536
9	84	504	2520	10080	30240	60480		103908
10	120	840	5040	25200	100800	302400	604800	1039200

Example 3.4. *We tabulate the values of* $C_4(n,k), C_4(n)$ *for* $5 \leq n \leq 10, 3 \leq k \leq 8$.

n \ k	3	4	5	6	7	8	$C_4(n)$
5	5						5
6	15	30					45
7	35	105	210				350
8	70	280	840	1680			2870
9	126	630	2520	7560	15120		25956
10	210	1260	6300	25200	151200	302400	486570

Example 3.5. *We tabulate the values of* $C_5(n,k), C_5(n)$ *for* $6 \leq n \leq 10, 3 \leq k \leq 7$.

n \ k	3	4	5	6	7	$C_4(n)$
6	6					6
7	21	42				63
8	56	168	336			560
9	126	504	1512	3024		5166
10	252	1260	5040	15120	30240	51912

4. Maximal Elements in the Set of All Chain Topologies on Finite Set

In this section, we construct the family of all super topologies of partial chain topologies with respect to any set. Moreover, we prove that for any partial chain topology with respect to any set, there exists chain family, having this partial chain topology as a minimum element, providing all maximal elements in the set of all chain topologies on finite set, and so we find the number of them.

Theorem 4.1. Let X be the set having n elements. For each $\tau_A \in c_A(n,k)$, there exists $\Phi_A^s \subset c_A(n,k+s)$, such that $n-|A|-k+2 \geq s \geq 0$, and

(i) $\tau_A \subset \tau_A^s$, for each $\tau_A^s \in \Phi_A^s$.

(ii) $\Phi_A^0 = \{\tau_A\}, |\Phi_A^s| = P(n-|A|-k+3, s)$.

(iii) For each $\tau_A^{s_1} \in \Phi_A^{s_1}$, there exists $\tau_A^{s_2} \in \Phi_A^{s_2}$, for each $s_2 > s_1$.

Proof.

(i) Let

$$\tau_A = \{X, \phi, A = U_0, U_1, U_2, U_3, U_1, \ldots., U_{k-3}\} \in c_A(n,k).$$

Then

$$\tau_A \subset \tau_A^s = \tau_A \cup \{U_{k-2}, U_{k-1}, \ldots\ldots, U_{k-3+s}\} \in c_A(n,k+s),$$

For each s such that $n-|A|-k+2 \geq s \geq 1$ and $\tau_A^0 = \tau_A$.

Taking Φ_A^s to be the family $\{\tau_A^s : n-|A|-k+2 \geq s \geq 0\}$, completes the proof of (i).

(ii) Since, we can choose U_{k-2} with $n-|A|-k+3$ methods, U_{k-1} with $n-|A|-k+2$ methods, and so we can choose U_{k-3+s} with $n-|A|-k-s+4$ methods. Then by using the principle of counting rule, it follows that the number of elements of Φ_A^s,

$$|\Phi_A^s| = (n-|A|-k+3)(n-|A|-k+3)\ldots(n-|A|-k-s+4) = P(n-|A|-k+3, s)$$

(iii) Since,

$$\tau_A^{s_1} = \tau_A \cup \{U_{k-2}, U_{k-1}, \ldots\ldots, U_{k-3+s_1}\}$$

and

$$\tau_A^{s_2} = \tau_A \cup \{U_{k-2}, U_{k-1}, \ldots\ldots, U_{k-3+s_2}\}$$

Then $s_2 > s_1$, implies that $\tau_A^{s_2} = \tau_A^{s_1} \cup \{U_{k-3+s_{1+t}} : s_2 - s_1 > t > 0\}$

Theorem 4.2. For each $\tau_A \in c_A(n,k)$, there exists the family $\psi_{\tau_A}^s$, $n-|A|-k+2 \geq s \geq 0$, such that $\psi_{\tau_A}^0 = \{\{\tau_A\}\}$, and the following two conditions are satisfied:

$\Lambda^s \in \psi_{\tau_A}^s \Rightarrow \Lambda^s$ is chain of partial chain topologies with respect to A, with respect to the inclusion relation, having τ_A as a minimum element.

$$\left|\psi_{\tau_A}^s\right| = \prod_{t=1}^{s} P(n-|A|-k+3, t)$$

Proof

(i) We choose $\Lambda^0 = \{\tau_A\}$, and we construct the family Λ^s, where $n-|A|-k+2 \geq s \geq 1$ as follows:

The first element belongs to Λ^s is τ_A, the second element is $\tau_A^1 \in \Phi_A^1$, the third element is $\tau_A^2 \in \Phi_A^2$, where $\tau_A^1 \subset \tau_A^2$, and so the th^t element is $\tau_A^{t-1} \in \Phi_A^{t-1}$, where $\tau_A^{t-2} \subset \tau_A^{t-1}$, $s \geq t \geq 4$.

Hence, $\Lambda^s = \{\tau_A, \tau_A^1, \tau_A^2, \ldots\ldots., \tau_A^s\}$ is chain with respect to the inclusion relation, having τ_A as the minimum element.

Taking $\psi_{\tau_A}^s$ to be the family of all Λ^s, completes the proof of (i).

(ii) Since $|\Phi_A^s| = P(n-|A|-k+3, s)$, then by using the principle counting rule

$$\left|\psi_{\tau_A}^s\right| = \prod_{t=1}^{s} P(n-|A|-k+3, t), n-|A|-k+2 \geq s \geq 1$$

Theorem 4.3. Let X be the set having n elements, then $\tau_{\{x\}} \in c_{\{x\}}(n, n+1)$; $x \in X$ is a maximal element in the set of all partial chain topologies on X.

Proof. Since $k = n+1, |A| = 1, n-|A|-k+2 \geq s \geq 0$, then $s = 0$, and so $\Phi_{\{x\}}^s = \Phi_{\{x\}}^0 = \{\tau_{\{x\}}\}$. Therefore, from the construction of the chain Λ^s, it follows that $\Lambda^s = \Lambda^0 = \{\tau_{\{x\}}\}$. Hence any chain of partial chain topologies with respect to $\{x\}$ with $n+1$ open set, with respect to the inclusion has only one element $\tau_{\{x\}}$. Implies $\tau_{\{x\}}$ is a maximal element in the set of all partial chain topologies.

Theorem 4.4. Let X be the set having n elements, then the number of all maximal elements in the set of all chain topologies on X equal $n!$.

Proof. Since the family of all maximal elements in the set of all chain topologies on X is $\{\tau_{\{x\}} \in c_{\{x\}}(n, n+1) : x \in X\}$, and since

$$C_r(n,k) = \frac{P(n, k+r-3)}{r!}, \text{ where } n-r+2 \geq k \geq 3, n-1 \geq r \geq 1.$$

Then the number of all maximal elements in the set of all chain topologies on X is

$$C_1(n, n+1) = \frac{P(n, n+1+1-3)}{1!} = P(n, n-1) = n!$$

5. Conclusion

Although, there are lots of equivalent ways of formulating for computing the number of topological spaces in finite set. In this paper, we formulated special case for computing the number of chain topological spaces, and maximal elements with the natural generalization.

(1) Computing the number of all partial chain topologies

with respect to the given subset A, providing the number of all chain topologies on it.

(2) Constructing all super topologies of partial chain topologies with respect to any set, providing all maximal elements in the set of all chain topologies on finite set, and so we find the number of them.

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