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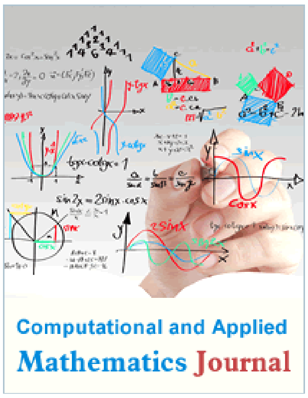

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New Types of Strongly Functions and Quasi Functions in Topological Spaces via e-Open Sets

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Abstract
The purpose of this paper is to introduce and investigate several new classes of functions called, e-open, e-closed, quasi e-open, quasi e-closed, strongly e-open and strongly e-closed functions in topological spaces by using the concept of e-open sets. Several new characterizations and fundamental properties concerning of these new types of functions are obtained. Furthermore, these kinds of functions have strong application in the area of image processing and have very important applications in quantum particle physics, high energy physics and superstring theory.

1. Introduction

Several generalized forms of open and closed functions, strongly functions and quasi functions in topological spaces have been introduced and investigated over the course of years. Certainly, it is hard to say whether one form is more or less important than another. Functions and of course open and closed functions, strongly functions and quasi functions stand among the most important and most researched points in the whole of mathematical science. Various interesting problems arise when one considers openness and closeness. Its importance is significant in various areas of mathematics and related sciences. In 2008, Erdal Ekici [1] introduced a new class of generalized open sets called e-open sets and studied several fundamental and interesting properties of e-open sets and introduced a new class of continuous functions called e-continuous functions into the field of topology. In this paper, we will continue the study of related functions by involving e-open sets. The aim of this paper is to introduce and investigate several new types of open and closed functions, strongly functions and quasi functions in topological spaces via e-open sets. Some characterizations and several interesting properties of these functions are discussed. Additionally, these kinds of functions have strong application in the area of Image Processing and have very important applications in quantum particle physics, high energy physics and superstring theory.

2. Preliminaries

Throughout the present paper, (X, T) and (Y, T^*) (or simply X and Y) mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset A of X, The closure and interior of A are denoted by Cl(A) and Int(A), respectively. We recall the following definitions, which will be used often throughout this paper.

A subset A of a space (X, T) is called δ-open [2] if for each $x \in A$ there exists a regular open set V such that $x \in V \subset A$. The δ-interior of A is the union of all regular open sets

Contained in A and is denoted by $Int_\delta(A)$. The subset A is called δ-open [2] if $A = Int_\delta(A)$. A point $x \in X$ is called a δ-cluster points of A [2] if $A \cap Int(Cl(V)) \neq \emptyset$ for each open set V containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $Cl_\delta(A)$. If $A = Cl_\delta(A)$), then A is said to be δ-closed [2]. The complement of δ-closed set is said to be δ-open set.

A subset A of a space X is called e-open [1] if $A \subset Cl(Int_\delta(A)) \cup Int(Cl_\delta(A))$. The complement of an e-open set is called e-closed. The intersection of all e-closed sets containing A is called the e-closure of A [1] and is denoted by e-Cl(A). The union of all e-open sets of X contained in A is called the e-interior [1] of A and is denoted by e-Int(A). The family of all e-open (resp. e-closed) subsets of X containing a point $x \in X$ is denoted by $E\Sigma(X, x)$ (resp. $EC(X, x)$. The family of all e-open (resp. e-closed) sets in X are denoted by $E\Sigma(X, T)$ (resp. $EC(X, T)$).

3. Characterizations of e-Open and e - Closed Functions

In this section, we obtain some characterizations and several properties concerning e-open functions and e-closed functions via e-open and e-closed sets.

Definition 3.1. A function $f: (X, T) \to (Y, T^*)$ is said to be e-open if $f(U) \in E\Sigma(Y, T^*)$ for every open set U in X.

Theorem 3.1. A function $f: (X, T) \to (Y, T^*)$ is e-open if and only if for each $x \in X$ and each open set U in X with $x \in U$, there exists a set $V \in E\Sigma(Y, T^*)$ containing $f(x)$ such that $V \subset f(U)$.

Proof: The proof is follows immediately from definition (3.1).

Theorem 3.2. Let $f: (X, T) \to (Y, T^*)$ be e-open. If $V \subset Y$ and M is a closed subset of X containing $f^{-1}(V)$, then there exists a set $F \in EC(Y, T^*)$ containing V such that $f^{-1}(F) \subset M$.

Proof: Let $F = Y - f(X - M)$. Then, $F \in EC(Y, T^*)$, since $f^{-1}(V) \subset M$, we have, $f(X - M) \subset (Y - V)$ and so $V \subset F$. Also $f^{-1}(F) = X - f^{-1}[f(X - M)] \subset X - (X - M) = M$.

Theorem 3.3. A function $f: (X, T) \to (Y, T^*)$ is e-open if and only if $f[Int(A)] \subset$ e-Int$[f(A)]$, for every $A \subset X$.

Proof: Let $A \subset X$ and $x \in Int(A)$. Then there exists an open set U_x in X such that $x \in U_x \subset A$. Now $f(x) \in f(U_x) \subset f(A)$, Since f is e-open, $f(U_x) \in E\Sigma(Y, T^*)$. Then,

$f(x) \in$ e-Int$[f(A)]$. Thus $f[Int(A)] \subset$ e-Int$[f(A)]$.

(Conversely), Let U be an open set in X. Then by assumption, $f[Int(U)] \subset$ e-Int$[f(U)]$. Since e-Int$[f(U)] \subset f(U)$, $f(U) =$ e-Int$[f(U)]$. Thus $f(U) \in E\Sigma(Y, T^*)$. So f is e-open.

Remark 3.1. The equality in the theorem (3.3) need not be true as shown in the following example.

Example 3.1. Let X = Y = {1, 2}, and T be the indiscrete topology on X and T^* be the discrete topology on Y. Then we have $E\Sigma(X, T) = \{\emptyset, X, \{1\}, \{2\}\}$ and $E\Sigma(Y, T^*) = T^*$, Let $f: (X, T) \to (Y, T^*)$ be the identity function and A = {1}. Then $f[Int(A)] = \emptyset$ and e-Int$[f(A)] = \{1\}$.

Theorem 3.4. A function $f: (X, T) \to (Y, T^*)$ is e-open if and only if $Int[f^{-1}(B)] \subset f^{-1}[$e-Int$(B)]$ for every $B \subset Y$.

Proof: Let B be any subset of Y. Then $f[Int(f^{-1}(B))] \subset f[f^{-1}(B)] \subset B$.

But $f[Int(f^{-1}(B))] \in E\Sigma(Y, T^*)$ since $Int[f^{-1}(B)]$ is open in X and f is e-open. Hence, $f[Int(f^{-1}(B))] \subset$ e-Int(B). Therefore $Int[f^{-1}(B)] \subset f^{-1}[$e-Int$(B)]$.

(Conversely), Let A be any subset of X. Then $f(A) \subset Y$. Hence by assumption, we have, $Int(A) \subset Int[f^{-1}(f(A)] \subset f^{-1}[$e-Int$(f(A))]$. Thus, $f[Int(A)] \subset$ e-Int$[f(A)]$, for every $A \subset X$. Hence, by Theorem (3.3), f is e-open.

Theorem 3.5. A function $f: (X, T) \to (Y, T^*)$ is e-open if and only if $f^{-1}[$e-Cl$(B)] \subset Cl[f^{-1}(B)]$ for every $B \subset Y$.

Proof: Suppose that f is e-open and $B \subset Y$ and let $x \in f^{-1}[$e-Cl$(B)]$. Then, $f(x) \in$ e-Cl(B). Let U be an open subset in X such that $x \in U$. since f is e-open, then $f(U) \in E\Sigma(Y, T^*)$ .Therefore $B \cap f(U) \neq \emptyset$. Then, $U \cap f^{-1}(B) \neq \emptyset$.

Hence $x \in Cl[f^{-1}(B)]$. Therefore we have $f^{-1}[$e-Cl$(B)] \subset Cl[f^{-1}(B)]$.

(Conversely), Let $B \subset Y$, then $(Y–B) \subset Y$. By assumption, $f^{-1}[$e-Cl$(Y– B)] \subset Cl[f^{-1}(Y – B)]$ this implies, $X–Cl[f^{-1}(Y – B)] \subset X – f^{-1}[$e-Cl$(Y – B)]$. Hence $X–Cl[X– f^{-1}(B)] \subset f^{-1}[(Y –$ e-Cl$(Y–B))]$. Now $X–Cl[X– f^{-1}(B)] = Int[X– (X– f^{-1}(B)] = Int[f^{-1}(B)]$.Then, we have $Y –$ e-Cl$(Y – B) =$ e-Int$[Y – (Y – B)] =$ e-Int(B). Then $Int[f^{-1}(B)] \subset f^{-1}[$e-Int$(B)]$. By Theorem (3.4) we have f is e-open.

Now we introduce some characterizations concerning e-closed functions.

Definition 3.2. A function $f: (X, T) \to (Y, T^*)$ is said to be e-closed if $f(M) \in EC(Y, T^*)$ for every closed set M in X.

Theorem 3.6. A function $f: (X, T) \to (Y, T^*)$ is e-closed if and only if e-Cl$[f(A)] \subset f[Cl(A)])]$ for every $A \subset X$.

Proof: Let f be e-closed function and let A be any subset of X. Then $f[Cl(A)] \in EC(Y, T^*)$. But $f(A) \subset f[Cl(A)]$. Then e-Cl$[f(A)] \subset f[Cl(A)]$.

(Conversely), Let A be a closed subset of X. Then by assumption, e-Cl$[f(A)] \subset f[Cl(A)] = f(A)$. This shows that $f(A) \in EC(Y, T^*)$. Hence f is e-closed.

Corollary 3.1. Let $f: (X, T) \to (Y, T^*)$ be e-closed and let $A \subset X$. Then, e-Int[e-Cl$(f(A)] \subset f[Cl(A)]$.

Theorem 3.7. Let $f: (X, T) \to (Y, T^*)$ be a surjective function. Then f is e-closed if and only if for each subset B of Y and each open set U in X containing $f^{-1}(B)$,there exists a set $V \in E\Sigma(Y, T^*)$ containing B such that $f^{-1}(V) \subset U$.

Proof: Let $V = Y – f(X – U)$, Then $V \in E\Sigma(Y, T^*)$. Since $f^{-1}(B) \subset U$, then we have $f(X – U) \subset Y– B$ so $B \subset V$. Also, $f^{-1}(V) = X – f^{-1}[f(X – U)] \subset X – (X – U) = U$.

(Conversely), Let M be a closed set in X and $y \in Y – f(M)$.

Then, $f^{-1}(y) \subset X – f^{-1}(f(M)) \subset X – M$ and $X – M$ is open in X. Hence by assumption, there exists a set $V_y \in E\Sigma(Y, y)$ such that $f^{-1}(V_y) \subset X–M$. This implies that $y \in V_y \subset Y– f(M)$. Thus $Y– f(M) = \bigcup \{V_y: y \in Y– f(M)\}$.

Hence $Y– f(M) \in E\Sigma(Y, T^*)$. Thus $f(M) \in EC(Y, T^*)$.

Definition 3.3. A function $f: (X, T) \to (Y, T^*)$ is said to be e-continuous [1], if $f^{-1}(V)$ is e-open in X for every open set V of Y.

Theorem 3.8. Let $f: X \to Y$ be a bijective. Then the Following are equivalent:

(a) f is e-closed; (b) f is e- open; (c) f^{-1} is e-continuous.

Proof: (a) $\Rightarrow$ (b): Let U be an open subset of X. Then X–U is closed in X. By (a), f(X – U)$\in$ EC(Y, T*). But f(X–U) = f(X) – f(U) = Y – f(U). Thus f(U)$\in$ EΣ(Y, T*).

(b) $\Rightarrow$ (c): Let U be an open subset of X. Since f is e-open. Then, $f(U) = (f^{-1})^{-1}(U) \in$ EΣ(Y, T*).

Hence f^{-1} is e-continuous.

(c) $\Rightarrow$ (a): Let M be an arbitrary closed set in X. Then X–M is open in X. Since f^{-1} is e-continuous, then $(f^{-1})^{-1}$(X–M) $\in$ EΣ(Y, T*). But $(f^{-1})^{-1}$(X–M) = f(X–M) = Y – f(M), thus f(M) $\in$ EC(Y, T*).

Definition 3.4. A space (X, T) is said to be:

a) e-T_1 [3] if for each pair of distinct points x and y of X, there exist e-open sets A and B containing x and y, respectively, such that x$\notin$ B and y$\notin$ A.

b) e-T_2 [3] if for each pair of distinct points x and y in X, there exist disjoint e-open sets A and B in X such that x$\in$ A and y$\in$ B.

Theorem 3.9. If f: (X, T) $\rightarrow$ (Y, T*) is e-open bijection. Then the following hold:

(a) If X is T_1 then Y is e-T_1. (b) If X is T_2 then Y is e-T_2.

Proof: (a) - Let y_1 and y_2 be any distinct points in Y. Then there exist x_1 and x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is T_1 then, there exist two open sets U and V in X with $x_1\in$ U, $x_2\notin$ U and $x_2\in$ V, $x_1\notin$ V. Now f(U) and f(V) are e-open in Y with $y_1\in f$(U), $y_2\notin f$(U) and $y_2\in f$(V), $y_1\notin f$(V).

Proof: (b) - is similar to (a). Thus is omitted.

Definition 3.5. A space (X, T) is said to be:

a) e-compact [4] if every cover of X by e-open sets has a finite sub cover.

b) e-Lindelof if every cover of X by e-open sets has a countable subcover.

Theorem 3.10. If f: (X, T) $\rightarrow$ (Y, T*) is e-open bijective. Then the following properties are hold:

(a) If Y is e-compact, then X is compact. (b) If Y is e-Lindelof, then X is Lindelof.

Proof: (a) - Let $U_1 = \{U_\lambda: \lambda\in \Delta\}$ be an open cover of X. Then $K_1 = \{f(U_\lambda): \lambda\in \Delta\}$ is a cover of Y by e-open sets in Y. Since Y is e-compact, Then K_1 has a finite subcover $K_2 = \{f(U_{\lambda 1}), f(U_{\lambda 2}), \ldots, f(U_{\lambda n})\}$ for Y. Then $U_2 = \{U_{\lambda 1}, U_{\lambda 2}, \ldots, U_{\lambda n}\}$ is a finite subcover of U for X.

Proof: (b): is similar to (a). Thus is omitted.

Definition 3.6. A space (X, T) is said to be e-connected [3] if X cannot be written as the union of two nonempty disjoint e-open sets.

Theorem 3.11. If a function f: (X, T) $\rightarrow$ (Y, T*) is an e-open surjective and Y is e-connected. Then X is connected.

Proof: Suppose that X is not connected. Then there exist two non-empty disjoint open sets U and V in X such that X = U$\cup$V. Then f(U) and f(V) are non-empty disjoint e-open sets in Y with Y = f(U) $\cup$ f(V) which contradicts the fact that Y is e-connected.

4. Characterizations of Quasi e - Open Functions

In this section, we obtain some characterizations and several properties concerning quasi e-open functions via e-open sets.

Definition 4.1. A function f: (X, T) $\rightarrow$ (Y, T*) is said to be quasi e-open if the image of every e-open set in X is open in Y.

Remark 4.1. (a) It is clear that, the concepts quasi e-openness and e-continuity coincide if the function is a bijection.

(b) It is obvious that, every quasi e-open function is open as well as e-open. However, the converses of the implications are not true in general as shown in the following example.

Example 4.1. Let X = Y = {1, 2, 3}, define a topology T= {Ø, X, {1}, {2, 3}} Then the identity function f: (X, T) $\rightarrow$ (Y, T*) is e-open as well as open but not quasi e-open.

Definition 4.2. A subset A is called an e-neighborhood of a point x in X if there exists an e-open set U such that x$\in$ U$\subset$ A.

Theorem 4.1. For a functions f: (X, T) $\rightarrow$ (Y, T*) the following properties are equivalent:

a) f is quasi e-open;

b) For each subset A of X, f[e-Int(A)] $\subset$ Int[f(A)];

c) For each x$\in$ X and each e-neighborhood U of x in X, there exists a neighborhood V of f(x) in Y such that V$\subset$ f(U).

Proof: (a) $\Rightarrow$ (b). Let f be quasi e-open and A $\subset$ X. Now we have Int(A)$\subset$A and e-Int(A)$\in$ EΣ(X, T). Hence we obtain that f [e-Int(A)] $\subset$ f (A). Since f [e-Int (A)] is open, then f [e-Int(A)] $\subset$ Int[f (A)]. (b) $\Rightarrow$ (c). Let x$\in$ X and U be an e-neighborhood of x in X. Then there exists V$\in$ EΣ(X, T) such that x$\in$ V$\subset$ U. Then by (b), we have, f(V) = f[e-Int(V)] $\subset$ Int[f(V)] and hence f(V) = Int[f(V)]. Therefore, it is follow that f(V) is open in Y such that f(x)$\in$ f(V)$\subset$$f$(U).

(c) $\Rightarrow$ (a). Let U$\in$ EΣ(X, T). Then for each y$\in$ f(U), there exists a neighborhood V_y of y in Y such that $V_y \subset f$(U). Since V_y is a neighborhood of y, there exists an open set W_y in Y such that y$\in$ $W_y \subset V_y$. Thus, f(U) = $\bigcup\{W_y$: y$\in$ f(U)$\}$ which is an open set in Y. This implies that f is quasi e-open function.

Theorem 4.2. A function f: (X, T) $\rightarrow$ (Y, T*) is quasi e-open if and only if e-Int[f^{-1}(B)] $\subset$ f^{-1}[Int(B)] for every subset B of Y .

Proof: Let B be any subset of Y. Then, e-Int[f^{-1}(B)] $\in$ EΣ(X, T) and f is quasi e-open, then f[e-Int(f^{-1}(B))] $\subset$ Int[f(f^{-1}(B))] $\subset$ Int(B). Thus, e-Int[f^{-1}(B)] $\subset$ f^{-1}[Int(B)]. (Conversely), Let U$\in$ EΣ(X, T). Then by assumption e-Int[f^{-1}(f(U))] $\subset$ f^{-1}[Int(f(U))] then, e-Int(U)$\subset$ f^{-1}[Int(f(U))], but e-Int(U) = U so U$\subset$ f^{-1}[Int(f(U))] and hence f(U)$\subset$ Int(f(U) so f is quasi e-open.

Theorem 4.3. A function f: (X, T) $\rightarrow$ (Y, T*) is quasi e-open if and only if for any subset B of Y and for any set M$\in$ EC(X, T) containing f^{-1}(B), there exists a closed subset F of Y containing B such that f^{-1}(F)$\subset$M.

Proof: Let f be quasi e-open and B$\subset$Y. Let M $\in$ EC(X, T) with f^{-1}(B)$\subset$M. Now, put F = Y – f(X–M). It is clear that since f^{-1}(B)$\subset$M, B$\subset$F. Since f is quasi e-open, F is a closed subset of Y. Also, we have f^{-1}(F)$\subset$M.

(Conversely), Let U$\in$ EΣ(X, T) and put B = Y–f(U). Then X–U$\in$ EC(X, T) with f^{-1}(B)$\subset$X–U. By assumption, there exists a closed set F of Y such that B$\subset$F and f^{-1}(F)$\subset$X – U. Hence, we obtain f(U)$\subset$Y–F. On the other hand, it follows that B ⊂ F, Y–F ⊂ Y–B = f (U). Thus, we have f (U) = Y–F which is open and hence f is a quasi e-open function.

Theorem 4.4. A function f: (X, T) → (Y, T^*) is quasi e-open if and only if f^{-1}[Cl(B)] ⊂ e-Cl[f^{-1}(B)] for every subset B of Y.

Proof: Suppose that f is quasi e-open function. For any subset B of Y, f^{-1}(B) ⊂ e-Cl[f^{-1}(B)]. Therefore by Theorem (4.3), there exists a closed set F in Y such that B ⊂ F and f^{-1}(F) ⊂ e-Cl[f^{-1}(B)]. Therefore, we obtain, f^{-1}[Cl(B)] ⊂ f^{-1}(F) ⊂ e-Cl[f^{-1}(B)]. (Conversely), Let B be any subset of Y and M∈ EC(X, T) with f^{-1}(B) ⊂ M. Put F = Cl(B), then we have B ⊂ F and F is closed and f^{-1}(F) ⊂ e-Cl[f^{-1}(B)] ⊂ M. Then by Theorem (4.3), the function f is quasi e-open.

Lemma 4.1. Let f: (X, T) → (Y, T^*) and g :(Y, T^*) → (Z, T^{**}) be two functions and gof: (X, T) → (Z, T^{**}) is quasi e-open. If g is continuous injective, then f is quasi e-open.

Proof: Let U be a e-open set in X, then (gof)(U) is open in Z since gof is quasi e-open. Again g is an injective continuous function, f (U) = g^{-1}(gof (U)) is open in Y. This shows that f is quasi e-open.

Theorem 4.5. If f: (X, T) → (Y, T^*) is quasi e-open bijective. Then the following properties are hold:

(a) If (X, T) is e-T_1 then (Y, T^*) is T_1. (b) If (X, T) is e-T_2 then (Y, T^*) is T_2.

Theorem 4.6. If f: (X, T) → (Y, T^*) is quasi e-open bijective. Then the following hold:

(a) If (Y, T^*) is compact, then (X, T) is e-compact. (b) If (Y, T^*) is Lindelof, then (X, T) is e-Lindelof.

Theorem 4.7. If f: (X, T) → (Y, T^*) is quasi e-open surjective and Y is Connected. Then X is e-connected.

Proof: The proofs of theorems {(4.5), (4.6), (4.7)} similar to the proofs of theorems {(3.9), (3.10), (3.11)} respectively.

Definition 4.3. A function f :(X, T) → (Y, T^*) is called pre-e-open if the image of each e-open set of X is an e-open set in Y.

Definition 4.4. A topological space (X, T) is said to be a T_e-space [5] if every e-open subset of (X, T) is open in (X, T).

Remark 4. 2. Let f :(X, T) → (Y, T^*) be a quasi e-open function. If Y is a T_e-space, then quasi e-openness coincide with pre-e-openness.

Definition 4.5. A function f :(X, T) → (Y, T^*) is said to be e-irresolute [3] if f^{-1}(V) is e-open in X for every e-open set V of Y.

Theorem 4.8. Let f: (X, T) → (Y, T^*) and g:(Y, T^*) → (Z, T^{**}) be two functions such that gof: (X, T) → (Z, T^{**}) is quasi e-open.

a) If f is e-irresolute surjective, then g is open.

b) If g is e-continuous injective, then f is pre-e-open.

Proof: (a) - Suppose that V ∈ EΣ(Y, T^*). Since f is e-irresolute, then f^{-1}(V) is e-open in (X, T). Since gof is quasi e-open and f is surjective, (gof (f^{-1}(V))) = g(V), which is open in (Z, T^{**}). This implies that g is an open function.

(b). Suppose that V∈ EΣ(X, T). Since gof is quasi e-open, (gof)(V) is open in (Z, T^{**}). Again g is a e-continuous injective function, g^{-1}(gof(V)) = f (V), which is e-open in (Y, T^*). This shows that f is pre e-open.

5. Characterizations of Quasi e - Closed Functions

In this section, we obtain some characterizations and several properties concerning quasi e-closed functions via e-closed sets.

Definition 5.1. A function f: (X, T) → (Y, T^*) is said to be quasi e-closed if the image of every e-closed set in X is closed in Y.

Remark 5.1. Clearly, every quasi e-closed function is closed as well as e-closed, but the converses of the implications are not true as shown in example (4.1).

Theorem 5.1. If a function f: (X, T) → (Y, T^*) is quasi e-closed. Then, f^{-1}[Int(B)] ⊂ e-Int[f^{-1}(B)] for every subset B of Y.

Proof: This proof is similar to the proof of theorem (4.2).

Theorem 5.2. A function f: (X, T) → (Y, T^*) is quasi e-closed if and only if for any subset B of Y and for any set U ∈ EΣ(X, T) containing f^{-1}(B), there exists an open subset V of Y containing B such that f^{-1}(V) ⊂ U.

Proof: This proof is similar to the proof of theorem (4.3).

Definition 5.2. A function f :(X, T) → (Y, T^*) is called pre-e-closed if the image of each e-closed set of (X, T) is an e-closed set in (Y, T^*).

Definition 5.3. A space X is said to be a C_e-space if every e-closed subset in X is closed in X.

Remark 5.2. Let f :(X, T) → (Y, T^*) be a quasi e-closed function. If Y is C_e-space, then quasi e-closedness coincides with pre-e-closedness.

Theorem 5.3. Let f :(X, T) → (Y, T^*) and g:(Y, T^*) → (Z, T^{**}) be any two functions. Then:

a) If f is quasi e-closed and g is quasi e-closed, then gof is quasi e-closed;
b) If f is e-closed and g is quasi e-closed, then gof is closed;
c) If f is quasi e-closed and g is e-closed, then gof is pre-e-closed;
d) If f is pre-e-closed and g is quasi e-closed, then gof is quasi e-closed.

Proof: The proof is obvious thus omitted.

Theorem 5.4. Let (X, T) and (Y, T^*) be topological spaces. Then the function f: (X, T) → (Y, T^*) is a quasi e-closed if and only f (X) is closed in Y and f (V)\ f (X \V) is open in f (X) whenever V is e-open in X.

Proof: Suppose f: (X, T) → (Y, T^*) is a quasi e-closed function. Since (X, T) is e-closed, f(X) is closed in (Y, T^*) and f(V)\ f(X\V) = f(X) \ f(X\V) is open in f(X) when V is e-open in (X, T). (Conversely), suppose f (X) is closed in (Y, T^*) , f (V)\ f(X\V) is open in f(X) when V is e-open in X, and let M be closed in X. Then, f(M) = f(X) \ (f(X\M) \ f(M)) is closed in f(X) and hence, closed in (Y, T^*).

Corollary 5.1. Let (X, T) and (Y, T^*) be topological spaces. Then a surjective function f: (X, T) → (Y, T^*) is quasi e-closed if and only if f (V)\ f (X\V) is open in (Y, T^*) whenever U is e-open in (X, T).

Corollary 5.2. Let (X, T) and (Y, T^*) be topological spaces and let f: (X, T) → (Y, T^*) be an e-continuous quasi e-closed surjective function. Then the topology on Y is {f (V)\ f (X\V):

V is e-open in X}.

Proof: Let H be open in Y. Then f^{-1}(H) is e-open in X, and $f(f^{-1}(H))\setminus f(X\setminus f^{-1}(H))$ = H. Hence, all open sets in Y are of the form $f(V)\setminus f(X\setminus V)$, V is e-open in X. On the other hand, all sets of the form $f(V)\setminus f(X\setminus V)$, V is e-open in X, are open in Y from corollary (5.1).

Definition 5.4. A topological space (X, T) e-normal [3] if for any pair of disjoint e-closed subsets F_1 and F_2 of X, There exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Theorem 5.5. Let (X, T) and (Y, T^*) be topological spaces with X is e-normal and let f: (X, T) → (Y, T^*) be an e-continuous quasi e-closed surjective function. Then Y is normal.

Proof: Let M_1 and M_2 be disjoint closed subsets of Y. Then $f^{-1}(M_1)$, $f^{-1}(M_2)$ are disjoint e-closed subsets of X. Since X is e-normal, there exist disjoint open sets V_1 and V_2 such that $f^{-1}(M_1) \subset V_1$ and $f^{-1}(M_2) \subset V_2$, Then $M_1 \subset f(V_1)\setminus f(X\setminus V_1)$ and $M_2 \subset f(V_2)\setminus f(X\setminus V_2)$. Further by Corollary (5.1), $f(V_1)\setminus f(X\setminus V_1)$ and $f(V_2)\setminus f(X\setminus V_2)$ are open sets in Y and clearly $(f(V_1)\setminus f(X\setminus V_1))\cap(f(V_2)\setminus f(X\setminus V_2)) = \emptyset$. This shows that Y is normal.

6. Characterizations of Strongly e - Open Functions

In this section, we obtain some characterizations and several properties concerning strongly e-open functions via e-open sets.

Definition 6.1. A function f: (X, T) → (Y, T^*) is said to be strongly e-open if $f(U) \in E\Sigma(Y, T^*)$ for each $U \in E\Sigma(X, T)$.

Theorem 6.1. Let f: (X, T) → (Y, T^*) and g:(Y, T^*) → (Z, T^{**}) be any two strongly e-open functions. Then *gof*: (X, T) → (Z, T^{**}) is strongly e-open function.

Proof: The proof is obvious thus omitted.

Theorem 6.2. A function f: (X, T) → (Y, T^*) is strongly e-open if and only if for each $x \in X$ and for each $U \in E\Sigma(X, T)$ with $x \in U$, there exists $V \in E\Sigma(Y, T^*)$ such that $f(x) \in V$ and $V \subset f(U)$.

Proof: It is obvious thus omitted.

Theorem 6.3. A function f: (X, T) → (Y, T^*) is strongly e-open if and only if for each $x \in X$ and for each e-neighborhood U of x in X, there exists an e-neighborhood V of $f(x)$ in Y such that $V \subset f(U)$.

Proof: Let $x \in X$ and let U be an e-neighborhood of x. Then there exists $H \in E\Sigma(X, T)$ such that $x \in H \subset U$. Then, $f(x) \in f(H) \subset f(U)$, since f is strongly e-open, Then $f(H) \in E\Sigma(Y, T^*)$. Hence $V = f(H)$ is an e-neighborhood of $f(x)$ and $V \subset f(U)$. (Conversely), Let $U \in E\Sigma(X, T)$ and $x \in U$, then U is an e-neighborhood of x. So by assumption, there exists an e-neighborhood $V_{f(x)}$ of $f(x)$ such that,

$F(x) \in V_{f(x)} \subset f(U)$. It follows that $f(U)$ is an e-neighborhood of each of its points. Therefore, $f(U) \in E\Sigma(Y, T^*)$, hence f is strongly e-open.

Theorem 6.4. A function f: (X, T) → (Y, T^*) is strongly e-open if and only if f[e-Int(A)] ⊂ e-Int[f(A)] for every A ⊂ X.

Proof: Let A ⊂ X and $x \in$ e-Int(A). Then there exists $U_x \in E\Sigma(X, T)$ such that $x \in U_x \subset A$. so $f(x) \in f(U_x) \subset f(A)$ and by assumption, $f(U_x) \in E\Sigma(Y, T^*)$. Hence,

$f(x) \in$ e-Int[f(A)]. Thus f[e-Int(A)] ⊂ e-Int[f(A)].

(Conversely), Let $U \in E\Sigma(X, T)$. Then by assumption, f[e-Int(U)] ⊂ e-Int[f(U)]. Since e-Int(U) = U and e-Int[f(U)] ⊂ f(U). Hence, f(U) = e-Int[f(U)]. Thus, $f(U) \in E\Sigma(Y, T^*)$.

Theorem 6.5. A function f: (X, T) → (Y, T^*) is strongly e-open if and only if e-Int[f^{-1}(B)] ⊂ f^{-1}[e-Int(B)] for every subset B of Y.

Proof: Let B ⊂ Y. Since e-Int[f^{-1}(B)] $\in E\Sigma(X, T)$ and f is strongly e-open, then f[e-Int(f^{-1}(B))] $\in E\Sigma(Y, T^*)$. Also we have f[e-Int(f^{-1}(B))] ⊂ f[f^{-1}(B)] ⊂ B. Hence, f[e-Int(f^{-1}(B))] ⊂ e-Int(B). Therefore, e-Int[f^{-1}(B)] ⊂ f^{-1}[e-Int(B)].

(Conversely), Let A be any subset of X. Then f(A) ⊂ Y. Hence by assumption, we obtain, e-Int(A) ⊂ e-Int[f^{-1}(f(A)] ⊂ f^{-1}[e-Int(f(A))].This implies that,

f[e-Int(A)] ⊂ f[f^{-1}(e-Int(f(A))] ⊂ e-Int[f(A)].Thus, f[e-Int(A)] ⊂ e-Int[f(A)], for all A ⊂ X. Hence, by Theorem (6.4), we obtain f is strongly e-open.

Theorem 6.6. A function f: (X, T) → (Y, T^*) is strongly e-open if and only if f^{-1}[e-Cl(B)] ⊂ e-Cl[f^{-1}(B)] for every subset B of Y.

Proof: Let B be any subset of Y and $x \in f^{-1}$[e-Cl(B)]. Then $f(x) \in$ e-Cl(B). Let $U \in E\Sigma(X, T)$ such that $x \in U$, by assumption, $f(U) \in E\Sigma(Y, T^*)$ and $f(x) \in f(U)$, Thus $f(U) \cap B \neq \emptyset$. Hence $U \cap f^{-1}(B) \neq \emptyset$. Therefore, $x \in$ e-Cl[f^{-1}(B)]. So we obtain, f^{-1}[e-Cl(B)] ⊂ e-Cl[f^{-1}(B)].

(Conversely), Let B ⊂ Y, then Y–B ⊂ Y, by assumption f^{-1}[e-Cl(Y–B)] ⊂ e-Cl[f^{-1}(Y–B)].This implies that, X–e-Cl[f^{-1}(Y–B)] ⊂ X–f^{-1}[e-Cl(Y–B)]. Hence,

X–e-Cl[X– f^{-1}(B)] ⊂ f^{-1}[Y–e-Cl(Y–B)]. Then, e-Int[f^{-1}(B)] ⊂ f^{-1}[e-Int(B)]. Now by Theorem (6.5), it follows that f is strongly e-open.

Theorem 6.7. Let f: (X, T) → (Y, T^*) be a function and g:(Y, T^*) → (Z, T^{**}) be a strongly e-open injective. If *gof*: (X, T) → (Z, T^{**}) is e-irresolute, then f is e-irresolute.

Proof: Let $U \in E\Sigma(Y, T^*)$. Since g is strongly e-open. Then, $g(U) \in E\Sigma(Z, T^{**})$. Also *gof* is e-irresolute, so we have $(gof)^{-1}[g(U)] \in E\Sigma(X, T)$. Since g is an injective, Therefore we have $(gof)^{-1}[g(U)] = (f^{-1} o g^{-1})[g(U)] = f^{-1}[g^{-1}(g(U))] = f^{-1}(U)$.Then, $f^{-1}(U) \in E\Sigma(X, T)$. So f is e-irresolute.

Theorem 6.8. Let f: (X, T) → (Y, T^*) be strongly e-open surjective and g:(Y, T^*) → (Z, T^{**}) be any function. If *gof*: (X, T) → (Z, T^{**}) is e-irresolute, then g is e-irresolute.

Proof: Let $V \in E\Sigma(Z, T^{**})$. Since *gof* is e-irresolute. Then, $(gof)^{-1}(V) \in E\Sigma(X, T)$. Also f is strongly e-open, so we have $f[(gof)^{-1}(V)] \in E\Sigma(Y, T^*)$. Since f is an surjective, Then, $f[(gof)^{-1}(V)] = [f o(gof)^{-1}](V) = [f o(f^{-1} o g^{-1})](V) = [(f o f^{-1}) o g^{-1}](V) = g^{-1}(V)$. Hence is g is e-irresolute.

Theorem 6.9. Let f: (X, T) → (Y, T^*) and g:(Y, T^*) → (Z, T^{**}) be two functions such that *gof*: (X, T) → (Z, T^{**}) is a strongly e-open function.

a) If f is e-irresolute surjective. Then, g is strongly e-open.

b) If g is e-irresolute injective. Then, f strongly e-open.

Proof: (a) - Let $U \in E\Sigma(Y, T^*)$. Since f is e-irresolute, $f^{-1}(U) \in E\Sigma(X, T)$. Now since *gof* is strongly e-open and f is surjective, then $(gof)(f^{-1}(U)) = g(U) \in E\Sigma(Z, T^{**})$. This implies that g is strongly e-open.

(b). Let V∈ EΣ(X, T). Since *gof* is strongly e-open, (*gof*)(V) ∈ EΣ(Z, T^{**}). Now since *g* is e-irresolute and injection, so $g^{-1}[(gof)(V)] = f(V)$∈ EΣ(Y, T^*). This shows that *f* is strongly e-open.

Theorem 6.10. Let *f*: (X, T) → (Y, T^*) be a strongly e-open bijective. Then the Following hold:

(a) If (X, T) is e-T_1 then (Y, T^*) is e-T_1. (b) If (X, T) is e-T_2 then (Y, T^*) is e-T_2.

Theorem 6.11. Let *f*: (X, T) → (Y, T^*) be a strongly e-open bijective. Then the Following hold:

a) If (Y, T^*) is e-compact, then (X, T) is e-compact.

b) If (Y, T^*) is e-Lindelof, then (X, T) is e-Lindelof.

Theorem 6.12. *f*: (X, T) → (Y, T^*) is a strongly e-open surjective and Y is e-connected then X is e-connected:

Proof: The proofs of theorems {(6.10), (6.11), (6.12)} similar to the proofs of theorems {(3.9), (3.10), (3.11)} respectively.

7. Characterizations of Strongly e - Closed Functions

In this section, we obtain some characterizations and several properties concerning strongly e-closed functions via e-closed sets.

Definition 7.1. A function f: (X, T) → (Y, T^*) is said to be strongly e-closed if *f*(M)∈ EC(Y, T^*) for each M∈ EC(X, T).

Theorem 7.1. Let *f*: (X, T) → (Y, T^*) and *g*:(Y, T^*) → (Z, T^{**}) be any two functions. Then:

a) If *f* is strongly e-closed and *g* is strongly e-closed, then *gof* is strongly e-closed;

b) If *f* is e-closed and *g* is strongly e-closed, then *gof* is e-closed;

c) If *f* is quasi e-closed and *g* is e-closed, then *gof* is strongly e-closed;

d) If *f* is strongly e-closed and *g* is quasi e-closed, then *gof* is quasi e-closed.

Proof: The proof is obvious thus omitted.

Theorem 7.2. A function *f*: X → Y is strongly e-closed if and only if e-Cl[*f*(A)] ⊂ *f*[e-Cl(A)] for every subset A of X.

Proof: Let *f* be strongly e-closed function and A ⊂ X. Then *f*[e-Cl(A)]∈ EC(Y, T^*). Since *f*(A) ⊂ *f*[e-Cl(A)], we obtain e-Cl[*f*(A)] ⊂ *f*[e-Cl(A)].

(Conversely), Let M∈ EC(X, T), by assumption, we obtain, *f*(M) ⊂ e-Cl[*f*(M)] ⊂ *f*[e-Cl(M)] = *f*(M). Hence *f*(M) = e-Cl[*f*(M)]. Thus, *f*(M)∈ EC(Y, T^*). It follows that *f* is strongly e-closed.

Theorem 7.3. Let *f*: (X, T) → (Y, T^*) be a strongly e-closed function and B ⊂ Y. If U∈ EΣ(X, T).with $f^{-1}(B) \subset U$, then there exists V∈ EΣ(Y, T^*) with B ⊂ V such that, $f^{-1}(B) \subset f^{-1}(V) \subset U$.

Proof: Let V = Y– *f*(X–U). Then Y–V = *f*(X–U). Since *f* is strongly e-closed, V∈ EΣ(Y, T^*).Since $f^{-1}(B) \subset U$, Then Y–V = *f*(X–U) ⊂ $f[f^{-1}(Y–B)] \subset$ Y–B. Hence, B ⊂ V. Also X–U ⊂ $f^{-1}[f(X–U)] = f^{-1}(Y–V) = X– f{-}1(V)$. So $f^{-1}(V) \subset U$.

Theorem 7.4. Let *f*: (X, T) → (Y, T^*) be a surjective strongly e-closed function and B, M ⊂ Y. If $f^{-1}(B)$ and $f^{-1}(M)$ have disjoint e-neighborhoods, then so have B and M.

Proof: Let F_1 and F_2 be the disjoint e-neighborhood of $f^{-1}(B)$ and $f^{-1}(M)$ respectively. Then by theorem (7.3) There exist two sets U, V∈ EΣ(Y, T^*). with B ⊂ U and M ⊂ V such that $f^{-1}(B) \subset f^{-1}(U) \subset$ e-Int(F_1) and $f^{-1}(M) \subset f^{-1}(V) \subset$ e-Int(F_2). Since F_1 and F_2 are disjoint, so are e-Int(F_1) and e-Int(F_2), and hence so $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint as well. Since *f* is a surjective function. Then it follows that U and V are disjoint too.

Theorem 7.5. A surjective function *f*: (X, T) → (Y, T^*) is strongly e-closed if and only if for each subset B of Y and each set U∈ EΣ(X, T) containing $f^{-1}(B)$, there exists a set V∈ EΣ(Y, T^*) containing B, such that $f^{-1}(V) \subset U$.

Proof: This follows from Theorem (7.3). (Conversely), Let M∈ EC(X, T) and y∈ Y– *f*(M), Then $f^{-1}(y) \in X– f^{-1}(f(M)) \subset$ X–M and X–M∈ EΣ(X, T). Hence by assumption, there exists a set V_y∈ EΣ(Y, y). Such that $f^{-1}(V_y) \subset$ X–M. This implies that y∈ V_y ⊂ Y– *f*(M). Thus, Y–*f*(M) = ⋃{V_y: y∈ Y– *f*(M)}. Hence, Y– *f*(M)∈ EΣ(Y, T^*).Therefore, *f*(M)∈ EC(Y, T^*).

Theorem 7.6. Let f: (X, T) → (Y, T^*) be a bijective. Then the following properties are equivalent:

(a) *f* is strongly e-closed; (b) *f* is strongly e-open; (c) f^{-1} is e-irresolute.

Proof: (a) ⇒ (b). Let U∈ EΣ(X, T), Then X–U∈ EC(X, T). Then By (a), *f*(X–U) ∈ EC(Y, T^*), but *f*(X–U) = *f*(X) – *f*(U) = Y– *f*(U). Thus *f*(U)∈ EΣ(Y, T^*).

(b) ⇒ (c). Let A ⊂ X, since *f* is strongly e-open, so by Theorem (6.6), $f^{-1}[\text{e-Cl}(f(A))] \subset \text{e-Cl}[f^{-1}(f(A))]$, It implies that e-Cl[*f*(A)] ⊂ *f*[e-Cl(A)]. Thus $\text{e-Cl}[(f^{-1})^{-1}(A)] \subset (f^{-1})^{-1}[\text{e-Cl}(A)]$, for all A ⊂ X. Then, it follows that f^{-1} is e-irresolute. (c) ⇒ (a). Let M∈ EC(X, T).Then X–M∈ EΣ(X, T). Since $f^{-1}$is e-irresolute, $(f^{-1})^{-1}$ (X–M)∈ EΣ(Y, T^*). But $(f^{-1})^{-1}$(X–M) = *f*(X–M) =Y–*f*(M). Thus *f*(M)∈ EC(Y, T^*).

Theorem 7.7. Let *f*: (X, T) → (Y, T^*) and *g*:(Y, T^*) → (Z, T^{**}) be two functions such that *gof*: (X, T) → (Z, T^{**}) is a strongly e-closed function. Then:

a) If *f* is e-irresolute and surjective. Then, *g* is strongly e-closed.

b) If *g* is e-irresolute and injective. Then, *f* is strongly e-closed.

Proof: Similar to the proof of theorem (6.9).

8. Conclusion

Generalized open and closed sets play a very a prominent role in general Topology and it applications. And many topologists worldwide are focusing their researches on these topics and this mounted to many important and useful results. Indeed a significant theme in General Topology, Real analysis and many other branches of mathematics concerns the variously modified forms of continuity, separation axioms etc by utilizing generalized open and closed sets. One of the well-known notions and that expected it will has a wide applying in physics and Topology and their applications is the notion of e-open sets. The importance of general topological spaces rapidly increases in both the pure and applied directions; it plays a significant role in data mining [6]. One can observe the influence of general topological spaces also in computer science and digital topology [7–9], computational topology for geometric and molecular design [10], particle physics, high energy physics, quantum physics, and Superstring theory [11–16,17]. In this paper we introduced and investigated the notions of new classes of functions which may have very important applications in quantum particle physics, high energy physics and superstring theory. Furthermore, the fuzzy topological version of the concepts and results introduced in this paper are very important. Since El-Naschie has shown that the notion of fuzzy topology has very important applications in quantum particle physics especially in related to both string theory and ε^{∞} theory.

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