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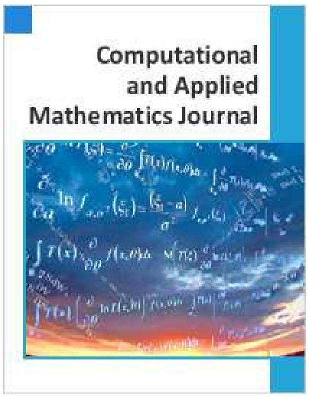

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On the Derivation of *p*-Radical Group in Group Theory

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Abstract
This paper introduces some derivation of p-radical group from the research of modern group theory. Through this a derivation of the presentation process, finally showed the methods and tools frequently used in the study of group theory, this is the purpose of this paper, because the deductive process is very useful to the professional and technical personnel of natural sciences, applied sciences and relevant professional, as well as their own research or study. Let H be arbitrary subgroup of group G, we define three sets on them, we define p-radical groups to use the properties of the three sets. Finally, we prove two interesting theorems about p-radical groups.

1. Introduction

Application of mathematical tools, almost every discipline to mature, must go through the process, especially the natural sciences, applied sciences, etc., but also inseparable from the mathematical tools, this paper introduces some derivation of p-radical group[1-5], the deductive process, the content is not necessarily a new thing, however, the reader, especially the natural sciences, applied sciences need mathematical tools such as professionals, can get a lot from the process of the mathematics method, thus, you can apply these methods to their own research or study.

The p-radical group is a kind of a special group in the Group theory[6-10], it is through the research of the special groups, to promote and enrich the content of Group theory.

About the study of the p-radical group, there have been at least a few decades of history, the researchers have been some results, and can be seen in some of the literature, such as [1], [2], [5] and [6], and in this paper, we give a brief introduction only.

2. Must Preparations and Lemmas

Such as a well-known, group theory research, can not leave the field theory, so the final result of our, intends to start from the field.

Since

$$J(F[G])=J(\overline{R}[G]\otimes_{\overline{R}} F) \tag{1}$$

for an extension field F of $\overline{R}$ it follows that $R(G)$ depends only on G and p.

First, for our proofs, we introduce some lemmas and give out the following

Definition 1 It is called $\sigma-$condition that for every S_p-subgroup P of group G,

$$\sum_{y\in P} a_{xy} = \sum_{y\in P} a_{yx} = 0 \tag{2}$$

for all $x \in G$.

Lemma 1 Let *H*, *A* be subgroups of *G* and let $\{x_i\}$ be a cross section of *H* in *G*. For any *R*[*H*] module *W* and any (*H*, *A*) double coset *D* define the *R*[*A*] module $W(D) = \oplus_{x_i \in D} W \otimes x_i$. Then $W(D) \approx \left\{W^x_{H^x \cap A}\right\}^A$ for any $x \in D$ and

$$(WG)A=\underset{D}{\oplus} W(D) \approx \underset{x}{\oplus} \left\{W^x_{H^x \cap A}\right\}^A \qquad (3)$$

where *D* and *HxA* range over all the (*H*, *A*) double cosets in *G*.

The proof of Lemma 1 can be found in Reference [1].

Lemma 2 Let *A*, *H* be subgroups of *G* and let *W* be an *R*[*A*] module and *V* a component of W^G. Suppose that $V_H = U_1 \oplus \dots \oplus U_t$ where each U_i is an indecomposable *R*[*H*] module. Then for each *i* there exists $x_i \in G$ such that U_i is $R[H \cap A^{x_i}]$-projective. In fact $U_i | \left\{W^{x_i}_{H \cap A^{x_i}}\right\}^H$.

Proof. It is immediate by Lemma 1.

Lemma 3 Let *V* be an indecomposable *R*[*G*] module with vertex *A*. Let *H* be a subgroup of *G* such that *V* is *R*[*H*]-projective and let $V_H = U_1 \oplus \dots \oplus U_t$ where each U_i is indecomposable with vertex A_i. Then

i. $A_i \subseteq_G A$ for each i.
ii. (ii) $V \mid U_j^G$ for some j and in this case $A_j =_G A$. Moreover if $A \subseteq H$ then $A_j =_H A$ for some j.
iii. If $A_j =_G A$ then V and U_j have a common source.

Proof. (i)By Lemma 2 U_i is $R[H \cap A^{x_i}]$-projective for suitable $x_i \in G$. Thus $A_i \subseteq_G H \cap A^{x_i} \subseteq_G A$.

(ii)Since *V* is *R*[*H*]-projective $V \mid (V_H)^G$, Thus $V \mid U_j^G$ for some *j* as *V* is indecomposable. Hence $A \subseteq_G A_j$ and so $A =_G A_j$ by (i). If moreover $A \subseteq H$, then $V \mid (V_A)^G$ and so $V \mid W^G$ for some indecomposable component *W* of $(U_j)_A$ for some *j*. Thus *W* has vertex *A*. By (i) applied to the components of $(U_j)_A$ we have $A \subseteq_H A_j$ and hence $A =_H A_j$ by (i).

(iii)If $A_j =_G A$ then $A_j = H \cap A^x$ for some $x \in G$. Thus $H \cap A^x$ is a vertex of U_j and $A^x \subseteq H$. Let *W* be an *R*[*A*] module which is a source of *V* then by *Lemma 2* $U_j \mid (W^x_{A^x})^H$. Thus W^x is a source of U_j.

Lemma 4 Let *H* be a subgroup of *G* and let *I* be a nilpotent left ideal of $\overline{R}[H]$. Let $I^x = x^{-1}Ix$ and let $\overline{I} = \bigcap_{x \in G} \overline{R}[G]I^x$. Then $\overline{I}$ is a nilpotent ideal of $\overline{R}$[G].

Proof. If $y \in G$ then $\overline{I}y \subseteq \bigcap_{x \in G} \overline{R}[G]\overline{I}^{xy} = \overline{I}$. Thus $\overline{I}$ is an ideal of $\overline{R}$[G]. Therefore $\overline{I}^{m+1} \subseteq \overline{I}^m(\overline{R}[G]I) = \overline{I}^m I$ for any integer $m \geq 1$. Hence by induction $\overline{I}^{m+2} \subseteq \overline{I}^2 I^m$ for any integer $m \geq 0$. Thus $\overline{I}$ is nilpotent.

Lemma 5 Let *e* be a centrally primitive idempotent in $\overline{R}[G]$ and let *B* be the block corresponding to *e*. The following are equivalent.

(i) H contains a defect group of B.
(ii) $VJ(\overline{R}[G])^n = VJ(\overline{R}[H])^n$ for all $\overline{R}[G]$ modules V in B and all integers $n \geq 0$.
(iii) $J(\overline{R}[G])e = J(\overline{R}[H])\overline{R}[G]e$.
(iv) $UJ(\overline{R}[G]) = UJ(\overline{R}[H])$ for all projective indecomposable $\overline{R}[G]$ modules U in B.

The proof of *Lemma 5* can be found in [2].

Lemma 6 Let *V* be an irreducible $\overline{R}[H]$ module[11-12]. Assume that every component of V^G lies in a block with a defect group which is contained in *H*. Then V^G is completely reducible.

Proof. By assumption there exist centrally primitive idempotents $\{e_i\}$ such that a defect group of each e_i lies in *H* and such that $V^G e = V^G$, where $e = \sum e_i$. By *Lemma 5*

$$\begin{aligned} V^G J(\overline{R}[G]) &= V^G J(\overline{R}[G])e = \left(V \underset{\overline{R}[H]}{\otimes} \overline{R}[G]\right) J(\overline{R}[G])e \\ &= V \underset{\overline{R}[H]}{\otimes} J(\overline{R}[G])e = V \underset{\overline{R}[H]}{\otimes} J(\overline{R}[H])\overline{R}[G]e \\ &= VJ(\overline{R}[H]) \underset{\overline{R}[H]}{\otimes} \overline{R}[G]e = (0) \end{aligned} \qquad (4)$$

Hence V^G is completely reducible.

Lemma 7 Suppose that *p* does not divide |G:H|. Let *V* be an irreducible $\overline{R}[H]$ module. Then V^G is completely reducible.

Proof. Since *H* contains a defect group of every block the result follows from *Lemma 6*.

*Lemma 8*Let *H*, *A* be subgroups of *G*. Let *V* be an *F*[*H*] module and let *W* be an *F*[*A*] module. Then

$$I(V^G, W^G) = \sum_x I(V^x_{H^x \cap A}, W_{H^x \cap A}) \qquad (5)$$

where *x* ranges over a complete set of (*H*, *A*) double coset representatives.

The proof of Lemma 8 can be found in Reference [1].

Lemma 9 Let *H* be a subgroup of *G*. Let *V* be an *F*[*G*] module and let *W* be an *F*[*H*] module. Then $I(W^G, V) = I(W, V_H)$ and $I(V, W^G) = I(V_H, W)$.

Proof. It is clear by *Lemma 8*.

Lemma 10 Suppose that $H \lhd G$. Let W be an $R[H]$ module. Let $\{x_i\}$ be a cross section of $T(W)$ in G. Then

$$(W^G)_H \approx |T(W):H|\left\{\underset{i}{\oplus} W^{x_i}\right\} \tag{6}$$

Proof. It is clear by Lemma 1.

Lemma 11 Let $H \lhd G$ and let V be an irreducible $F[G]$ module. Then there exists an irreducible $F[H]$ module W and an integer $e=e_H(V)$ such that $V_H \approx e\{\oplus W^{x_i}\}$ where x_i ranges over a cross section of $T(W)$ in G. Thus in particular V_H is completely reducible.

Proof. Let W be a minimal submodule of V_H. Then $Wx = W^x$ for $x \in G$ and so Wx is irreducible. Hence $\sum_{x\in G} W^x$ is completely reducible. Since $\sum W^x$ is sent into itself by multiplication by elements of $F[G]$ the irreducibility of V implies that $V_H = \sum W^x$. Thus if $\{x_i\}$ is a cross section of $T(W)$ in G it follows from Lemma 10 that $V_H \approx \oplus e_i W^{x_i}$. By Lemma 9

$$\begin{aligned} e_i I(W,W) &= e_i I(W^{x_i}, W^{x_i}) = I(V_H, W^{x_i}) \\ &= I(V,(W^{x_i})^G) = I(V, W^G) \end{aligned} \tag{7}$$

Hence $e = e_i$ is independent of i proving the result.

Let us begin to give out the properties of the three sets in the following.

Lemma 12 $R(G)=C(G)$.

Proof. Let $H \in R(G)$ and let V be an irreducible $\overline{R}[H]$ module. Then

$$VGJ(\overline{R}[G])=(V \underset{\overline{R}[H]}{\otimes} \overline{R}[G])J(\overline{R}[G])$$

$$=V \underset{\overline{R}[H]}{\otimes} J(\overline{R}[G]) \subseteq V \underset{\overline{R}[H]}{\otimes} J(\overline{R}[H])\overline{R}[G]$$

$$=VJ(\overline{R}[H])\ \overline{R}[G]=(0) \tag{8}$$

Thus V^G is completely reducible and so $H \in C(G)$. therefore $R(G) \subseteq C(G)$.

Suppose that $H \in C(G)$. Let $\{x_i\}$ be a cross section of H in G. Then an arbitrary element $a \in J(\overline{R}[G])$ is of the form $a=\sum_i a_i x_i$ with $a_i \in \overline{R}[H]$. If V is an irreducible $\overline{R}[H]$ module then $V^G a=0$ as V^G is completely reducible. Therefore $0=(V\otimes 1)a=\sum_i Va_i \otimes x_i$. Hence $Va_i = 0$ for all i. Hence each a_i annihilates every irreducible $\overline{R}[H]$ module and so $a_i \in J(\overline{R}[H])$for all i. Thus $a \in J(\overline{R}[H])\overline{R}[G]$. Hence $H \in R(G)$ and so $C(G) \subseteq R(G)$.

Lemma 13 $R(G) \subseteq D(G)$.

Proof. Let $H \in R(G)$. Let $V=In V_H(V)$ be a one dimensional $\overline{R}[H]$module. By Lemma 12 $H \in C(G)$ and so V^G is completely reducible. Thus $W| V^G$, where $W=Inv_G(W)$ and $\dim_{\overline{R}} W=1$. If Q is S_p-group of H then Q is a vertex of V. Let Pbe a S_P-group of G with $Q \subseteq P$. Similarly W has P as a vertex. Since $W|V^G$, we must have $P=Q$ by Lemma 3.

3. Definition of *p*-radical Group and the Relevant Solutions

Definition 2: A group G is a p-radical group if $R(G)=D(G)$ or equivalently $C(G)=D(G)$.

The next result shows in particular that G is p-radical if and only if the σ -condition is necessary and sufficient for an element of $\overline{R}(G)$ to be in the radical.

Now, let us give out our results mainly.

Theorem 1 The following conditions are equivalent.

(i) G is p-radical.

(ii) $R(G)$ contains a S_p-group of G.

(iii)

$$J(\overline{R}[G]) = \bigcap_{x\in G} J(\overline{R}[Px])\overline{R}[G] \tag{9}$$

where P is a S_p-group of G.

(iv)

$$AnnJ(\overline{R}[G]) = \sum_P \varepsilon_P\ \overline{R}[G] \tag{10}$$

where P ranges over all S_p-groups of G and $\varepsilon_P = \sum_{x\in P} x$.

(v) $\sum_{x\in G} a_x x \in J(F[G])$ if and only if $\sum_{y\in P} a_{xy} = \sum_{y\in P} a_{yx} = 0$ for all $x \in G$and every S_p-group of G.

Proof. (i)⇒(ii). It is clear by definition.

(ii)⇒(iii). Since $P \in R(G)$,$J(\overline{R}[G]) \subseteq J(R[P^x])\overline{R}[G]$ for all $x \in G$. Thus

$$J(\overline{R}[G]) \subseteq \bigcap_{x\in G} J(\overline{R}[Px])\overline{R}[G] \tag{11}$$

The opposite inclusion follows from *Lemma 4*.

(iii)⇒(i). By Lemma 13 it suffices to show that $D(G) \subseteq R(G)$. Let $H \in D(G)$ and let P be a S_p-group of G with $P \subseteq H$. By Lemma 4,

$$\bigcap_{x\in H} J(\overline{R}[Px])\overline{R}[H] \subseteq J(\overline{R}[H]) \tag{12}$$

Therefore,

$$\bigcap_{x\in G} J(\overline{R}[Px])\overline{R}[G] \subseteq \bigcap_{x\in G} J(\overline{R}[Px])\overline{R}[Hx]\overline{R}[G] \subseteq \bigcap_{x\in G} J(\overline{R}[Hx])\overline{R}[G] \tag{13}$$

By Lemma 4 $\bigcap_{x\in G} J(\overline{R}[H^x])\overline{R}[G] \subseteq J(\overline{R}[G])$. By assumption

$$J(\overline{R}[G])=\bigcap_{x\in G} J(\overline{R}[Px])\overline{R}[G] \quad (14)$$

Hence all these sets are equal and so

$$J(\overline{R}[G])=\bigcap_{x\in G} J(\overline{R}[Hx])\overline{R}[G]\subseteq J(\overline{R}[H])\overline{R}[G] \quad (15)$$

Therefore $H\in R(G)$ and so $D(G)\subseteq R(G)$.

(iii)⇔(iv). Since $J(\overline{R}[P])=\varepsilon_P \overline{R}[P]$, each of the statements follows from the other by taking annihilators.

(iv)⇔(v). Statement (v) is equivalent to the fact that a in $\overline{R}[G]$ is in $J(\overline{R}[G])$ if and only if $a\varepsilon_P = \varepsilon_P a=0$ for all S_p-groups P of G. In other words, $J(\overline{R}[G])=Ann(\sum_P \varepsilon_P \overline{R}[G])$. Thus (iv) and (v) can be derived from each other by taking annihilators.

4. Further Solutions

Corollary 1. Let P be a S_p-group of G and let $V=Inv_P V$ be an $\overline{R}$[G] module with $\dim_{\overline{R}} V=1$. Then G is p-radical if and only if V^G is completely reducible.

Proof. By Lemma 12 and Theorem 1 G is p-radical if and only if $P\in C(G)$. The result follows as up to isomorphism V is the unique irreducible $\overline{R}[P]$ module.

Theorem 2 Suppose that $H\lhd G$. Then the following hold.

(i) H∈D(G) if and only if H∈R(G).
(ii) If G is p-radical then G/H is p-radical.
(iii) If H is a p-group then G is p-radical if and only if G/H is p-radical.
(iv) If H∈D(G) then G is p-radical if and only H is p-radical.

Proof. Let P be a S_p-group of G.

(i) If H∈D(G) then H∈C(G) by Lemma 7. Thus H∈R(G) by Lemma 12. The converse follows form Lemma13.
(ii) Let V be an irreducible $\overline{R}$[PH/H] module. VG is completely reducible as an $\overline{R}$[G] module if and only if VG is completely reducible as an $\overline{R}$[G/H] module. The result follows from Corollary 1.
(iii) If G is p-radical then so is G/H by (ii). Suppose that G/H is p-radical. Let V be an irreducible $\overline{R}$[P] module. Since H⊆P, P is an irreducible $\overline{R}$[P/H] module and so VG is a completely reducible $\overline{R}$[G/H] module. Thus VG is completely reducible and G is p-radical by Corollary 1.
(iv) Let V be an irreducible $\overline{R}$[P] module. If VG is completely reducible then by Lemma 10 (VG)H is completely reducible. Thus by Lemma 1, VH is completely reducible. If VH is completely reducible then by Lemma 10 VG is completely reducible. The result follows from Corollary 1.

Corollary 2. Let $G=G_1\times G_2$. Then G is p-radical if and only if G_1 and G_2 are p-radical.

Proof. If G is p-radical then G_1 and G_2 are p-radical by Theorem 2 (ii). Suppose that G_1 and G_2 are p-radical. Let P_i be a S_p-group of G_i for i=1,2. By Theorem 2 $J(\overline{R}[G_i])\subseteq J(\overline{R}[P_i])\overline{R}[G_i]$. Since $J(\overline{R}[G])= J(\overline{R}[G_1])\overline{R}[G_2]+ J(\overline{R}[G_2])\overline{R}[G_1]$ as $\overline{R}[G]=\overline{R}[G_1]\otimes_{\overline{R}}\overline{R}[G_2]$ modules, it follows that

$$J(\overline{R}[G])\subseteq J(\overline{R}[P_1])\overline{R}[G]+ J(\overline{R}[P_2])\overline{R}[G].$$

Let $P=P_1\times P_2$. Then P is a S_p-group of G and $J(\overline{R}[P_i])\subseteq J(\overline{R}[P])$ for i=1,2. Thus $J(\overline{R}[G])\subseteq J(\overline{R}[P])\overline{R}[G]$. Thus $P\in R(G)$ and so G is p-radical by Theorem 1.

5. Conclusions

From the above derivation about p-radical group, we can see that in the study of modern group theory, the Sylow' subgroup always as a basis is important, module is often used in research of group, even in many branches of algebra, and is effective, and the nilpotent, indecomposable etc of group and ideal is also related to the study group, they often can make important properties.

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