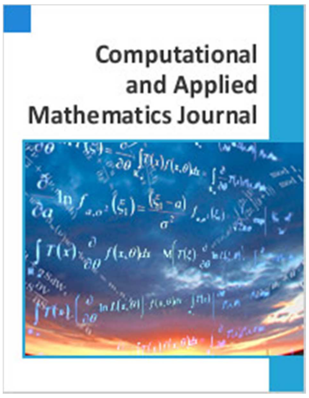
Computational and Applied Mathematics Journal
2016; 2(4): 34-37
http://www.aascit.org/journal/camj
ISSN: 2381-1218 (Print); ISSN: 2381-1226 (Online)

American Association for Science and Technology



Keywords
System of Difference Equations,
Positive Solution,
Periodicity,
Boundedness

Received: June 3, 2016
Accepted: June 15, 2016
Published: September 27, 2016

Behavior of Solutions of a System of Max-Type Difference Equations

Ali Gelişken

Mathematics Department, Kamil Özdağ Faculty of Science, Karamanoğlu Mehmetbey University, Karaman, Turkey

Email address
aligelisken@yahoo.com.tr, agelisken@kmu.edu.tr

Citation
Ali Gelişken. Behavior of Solutions of a System of Max-Type Difference Equations. *Computational and Applied Mathematics Journal.* Vol. 2, No. 4, 2016, pp. 34-37.

Abstract
We investigate the behavior of solutions of the system of the difference equations. $x_{n+1} = max\left\{A, \frac{C}{y_{n-k}}\right\}$, $y_{n+1} = max\left\{B, \frac{C}{x_{n-m}}\right\}$, $n = 0,1,2,\ldots$, where $k, m \in \mathbb{N}$ and A, B, C and the initial conditions $x_{-m}, x_{-m+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0$ are positive real numbers. We show that every solution of this system is bounded and eventually constsnt or eventually periodic with period *k+m+2*.

1. Introduction and Preliminaries

Mathematical model of a continuous event in engineering, physics, biology etc., is formed by using differantial equations. But, an incontinuous event can be determined by a difference equations. Also, difference equations are used to numerical solutions of differantial equations. So, there has been a great interest in studying difference equations and their systems. Some are in [1-15].

In [13], it was investigated the behavior of the positive solutions of the system of the difference equations

$$x_{n+1} = max\left\{\frac{A}{x_n}, \frac{y_n}{x_n}\right\}, \; y_{n+1} = max\left\{\frac{A}{y_n}, \frac{x_n}{y_n}\right\}. \tag{1}$$

In [4], it was investigated the periodic character of positive solutions of the system of the difference equations with max

$$x_{n+1} = max\left\{A, \frac{y_n}{x_{n-1}}\right\}, \; y_{n+1} = max\left\{B, \frac{x_n}{y_{n-1}}\right\}. \tag{2}$$

Motivated by these papers, it is investigated the behavior of the positive solution of,

$$x_{n+1} = max\left\{A, \frac{C}{y_{n-k}}\right\}, \; y_{n+1} = max\left\{B, \frac{C}{x_{n-m}}\right\}, \; n = 0,1,2,\ldots, \tag{3}$$

where $k, m \in \mathbb{N}$ and A, B, C and the initial conditions $x_{-m}, x_{-m+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0$ are positive real numbers.

Let $x_n = \sqrt{C} z_n$ and $y_n = \sqrt{C} w_n$ for $n \geq 0$. The change of variables reduces (3) to the following system of the difference equations

$$z_{n+1} = max\left\{a, \frac{1}{w_{n-k}}\right\}, \; w_{n+1} = max\left\{b, \frac{C}{z_{n-m}}\right\}, \; n = 0,1,2,\ldots, \tag{4}$$

where $a = \frac{A}{\sqrt{C}}$ and $b = \frac{B}{\sqrt{C}}$. So, we only investigate the solutions of (3) with $C = 1$.

In this study, we need the following definitions.

Definition 1. A sequence $\{x_n\}_{n=n_0}^{\infty}$ is said to be periodic with period p if there exists an integer $p \geq 1$ such that $x_{n+p} = x_n$ for $n \geq n_0$.

Definition 2. A sequence $\{x_n\}_{n=n_0}^{\infty}$ is said to be eventually periodic with period p if there exists an integer $N \geq n_0$ such that $\{x_n\}_{n=N}^{\infty}$ is periodic with period p, that is, $x_{n+p} = x_n$ for $n \geq N$.

Definition 3. A sequence $\{x_n\}_{n=n_0}^{\infty}$ is said to be bounded if there exists P and Q constants such that $P \leq x_n \leq Q$ for $n \geq n_0$.

2. Some Special Results

2.1. The System with $k = m = 0$

In this section, we consider the following system of difference equations with max

$$x_{n+1} = max\left\{A, \frac{1}{y_n}\right\}, y_{n+1} = max\left\{B, \frac{1}{x_n}\right\}, n = 0,1,2,..., \quad (5)$$

where A, B and x_0, y_0 initial conditions are positive real numbers.

Theorem 1. Every solution of (5) is eventually constant or eventually periodic with period 2. Moreover,

i). If $A \geq 1$ and $B \geq 1$, then every solution of (5) is eventually constant.

ii). If at least one of the parameters A, B is less than 1, then every solution of (5) is eventually constant or eventually periodic with period 2.

Proof.

i). For the purpose of the proof is more understandable the below cases, which can be united, consider individually. Firstly, we assume that $A = B = 1$. From the (5) we get immediately $x_n = 1$ and $y_n = 1$ for $n \geq 1$. Secondly, suppose that $A \geq 1$ and $B > 1$. Let $x_n = B^{z_n}$, $y_n = B^{w_n}$ for $n \geq 0$. The change of variables reduces (5) to the following system of the difference equations

$$z_{n+1} = max\{a, -w_n\},$$

$$w_{n+1} = max\{1, -z_n\},$$

$$n = 0,1,2,..., \quad (6)$$

where z_0, w_0 initial conditions are real numbers and $a = \log_B A$. Because of the selection of a, it is positive or equal to zero. From (6) we obtain

$$z_{n+2} = max\{a, min\{-1, z_n\}\} = a$$

and

$$w_{n+2} = max\{1, min\{-a, w_n\}\} = 1$$

for $n \geq 0$. Thirdly, we suppose $A > 1$ and $B \geq 1$. Let $x_n = A^{r_n}$ and $y_n = A^{s_n}$ for $n \geq 0$. The change of variables reduces (5) to the following system of the difference equations

$$r_{n+1} = max\{1, -s_n\}, \quad s_{n+1} = max\{b, -r_n\}, n = 0,1,\ldots, \quad (7)$$

where r_0, s_0 initial conditions are real numbers and $b = \log_A B$. Here, b is positive or equal to zero. From (7), we obtain

$$r_{n+2} = max\{1, min\{-b, r_n\}\} = 1$$

and

$$s_{n+2} = max\{b, min\{-1, s_n\}\} = b$$

for $n \geq 0$.

ii). Firstly, we assume that $A < 1$ and $B < 1$. The change of variables $x_n = C^{t_n}, y_n = C^{u_n}$ for $C > 1$ and $n \geq 0$, reduces (5) to the system

$$t_{n+1} = max\{c_1, -u_n\}, u_{n+1} = max\{c_2, -t_n\}, n = 0,1,\ldots, \quad (8)$$

where t_0, u_0 initial conditions are nonzero real numbers and $c_1 = \log_C A$, $c_2 = \log_C B$. Clearly, c_1 and c_2 are negative. From the (8) we have

$$t_{n+2} = max\{c_1, min\{-c_2, t_n\} \in \{c_1, -c_2, t_n\},$$

$$u_{n+2} = max\{c_2, min\{-c_1, u_n\}\} \in \{-c_1, c_2, u_n\}$$

and then we get

$$t_{n+3} = max\{c_1, -u_{n+2}\} = -u_{n+2},$$

$$u_{n+3} = max\{c_2, -t_{n+2}\} = -t_{n+2},$$

$$t_{n+4} = max\{c_1, -u_{n+3}\} = t_{n+2},$$

$$u_{n+4} = max\{c_2, -t_{n+3}\} = u_{n+2}$$

for $n = 0,1,\ldots$. It is easy to see that every solution of (8) is eventually constant if $t_{n+2} = -u_{n+2}$. So, we obtain that every solution of the system (8) is eventually constant or eventually periodic with period 2 in the case $A < 1$ and $B < 1$. Secondly, assume that $B < 1 < A$. Let $x_n = A^{r_n}$ and $y_n = A^{s_n}$ for $n \geq 0$. Then, the change of variables reduces the system (5) to the system (7) where r_0, s_0 initial conditions are nonzero real numbers and $b = \log_A B$. Here, b is negative. If $b \geq -1$, then we get

$$r_{n+2} = max\{1, min\{-b, r_n\} = 1$$

and

$$s_{n+2} = \max\{b, min\{-1, s_n\} = b$$

for $n \geq 0$ from (7). If $b < -1$, then we obtain

$$r_{n+2} = max\{1, min\{-b, r_n\}\} \in \{-b, 1, r_n\},$$

$$s_{n+2} = max\{b, min\{-1, s_n\}\} \in \{-1, b, s_n\},$$

$$r_{n+3} = max\{1, -s_{n+2}\} = -s_{n+2},$$

$$s_{n+3} = max\{1, -r_{n+2}\}\} = -r_{n+2},$$

$$r_{n+4} = max\{1, -s_{n+3}\} = r_{n+2},$$

$$s_{n+4} = max\{1, -r_{n+3}\} = s_{n+2}$$

for $n \geq 0$. Finally, suppose that $A < 1 < B$. Let $x_n = B^{z_n}, y_n = B^{w_n}$ for $n \geq 0$. Then, (5) implies (6) where z_0, w_0 initial conditions are nonzero real numbers and $a = \log_B A$. Here, a is negative. Similarly, we can obtain that every solution of (6) is eventually constant if $a \geq -1$. Also, it can be obtained that every solution of (6) is eventually constant or eventually periodic if $a < -1$. So, the proof is finished. ■

2.2. The System with $k = m = 1$

In this section, we consider the following system of the difference equations with maximum

$$x_{n+1} = max\left\{A, \frac{1}{y_{n-1}}\right\}, y_{n+1} = max\left\{B, \frac{1}{x_{n-1}}\right\}, n = 0,1,2, \ldots, \quad (9)$$

where A, B and x_0, x_{-1}, y_0, y_{-1} initial conditions are positive real numbers. By the change of variables

$x_n = C^{z_n}, y_n = C^{w_n}$ for $C > 1$ and $n \geq 0$, the system (9) is transformed into the following system of the difference equations

$$z_{n+1} = max\{a, -w_{n-1}\}, \; w_{n+1} = max\{b, -z_{n-1}\}, n = 0,1, \ldots, \quad (10)$$

where z_0, z_{-1}, w_0, w_{-1} initial conditions and the parameters $a = \log_C A$, $b = \log_C B$ are real numbers.

Theorem 2. Every solution of (9) is eventually constant or eventually periodic with period 4.

Proof. It is shown that every solution of (10) is eventually constant or eventually periodic with period 4. This is enough to prove the theorem. It is proved in following 2 cases.

Case 1. Suppose that $a \geq 0$ and $b \geq 0$, or $0 < -b < a$, or $0 < -a < b$. From (10), we get immediately

$$z_{n+2} = max\{a, min\{-b, z_{n-2}\}\} = a$$

and

$$w_{n+2} = max\{b, min\{-a, w_{n-2}\}\} = b,$$

for $n \geq 0$.

Case 2. Suppose that $a < 0$ and $b < 0$, or $0 < a < -b$, or $0 < b < -a$. From (10), we get

$$z_{n+2} = max\{a, min\{-b, z_{n-2}\}\} \in \{a, -b, z_{n-2}\},$$

$$w_{n+2} = max\{b, min\{-a, w_{n-2}\}\} \in \{-a, b, w_{n-2}\},$$

$$z_{n+3} = max\{a, min\{-b, z_{n-1}\}\} \in \{a, -b, z_{n-1}\},$$

$$w_{n+3} = max\{b, min\{-a, w_{n-1}\}\} \in \{-a, b, w_{n-1}\},$$

$$z_{n+4} = max\{a, -w_{n+2}\} = -w_{n+2},$$

$$w_{n+4} = max\{b, -z_{n+2}\}\} = -z_{n+2},$$

$$z_{n+5} = max\{a, -w_{n+3}\} = -w_{n+3},$$

$$w_{n+5} = max\{b, -z_{n+3}\}\} = -z_{n+3},$$

$$z_{n+6} = max\{a, -w_{n+4}\} = z_{n+2},$$

$$w_{n+6} = max\{b, -z_{n+4}\}\} = w_{n+2},$$

for $n > 0$. If $z_{n+2} = z_{n+3} = -w_{n+2} = -w_{n+3}$, then every solution of (10) is eventually constant. Otherwise, every solution of (10) is eventually periodic with period 4. So, the proof is finished. ■

3. Main Results

We consider the system of the difference equations

$$x_{n+1} = max\left\{A, \frac{1}{y_{n-k}}\right\},$$

$$y_{n+1} = max\left\{B, \frac{1}{x_{n-m}}\right\},$$

$$n = 0,1,2, \ldots, \quad (11)$$

where $k, m \in \mathbb{N}$ and A, B and the initial conditions $x_{-m}, x_{-m+1}, , \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0$ are positive real numbers.

Theorem 3. Every solution of (11) is bounded.

Proof. From the (11), we have $A \leq x_n$ and $B \leq y_n$ for $n \geq 1$. Using these results, we obtain

$$x_{n+k+1} = max\left\{A, \frac{1}{y_n}\right\}$$

$$\leq max\left\{A, \frac{1}{B}\right\}$$

and

$$y_{n+m+1} = max\left\{B, \frac{1}{x_n}\right\}$$

$$\leq max\left\{B, \frac{1}{A}\right\}$$

for $n \geq 1$. Then, we get immediately every solution of (11) is bounded. ■

Theorem 4. Every solution of (11) is eventually constant or eventually periodic with period $(k + m + 2)$.

Proof. Let $x_n = C^{z_n}, y_n = C^{w_n}$ and $C > 1$ for $n \geq 0$. Then, (11) implies the following systems of the difference equations

$$z_{n+1} = max\{a, -w_{n-k}\}, \; w_{n+1} = max\{b, -z_{n-m}\}, \text{n} = 0,1, \ldots, \quad (12)$$

where the initial conditions $z_{-m}, z_{-m+1}, , \ldots, z_0, w_{-k}, w_{-k+1}, \ldots, w_0$ and the parameters $a = \log_C A$, $b = \log_C B$ are real numbers. We prove every solution of (12) is eventually constant or eventually periodic with period (*k*+*m*+*2*). There are 2 cases which are possible.

Case 1. Suppose that $a \geq 0$ and $b \geq 0$, or $0 < -b < a$, or $0 < -a < b$. From (12), we get immediately

$$z_{n+k+2} = max\{a, min\{-b, z_{n-m}\}\} = a$$

and

$$w_{n+m+2} = max\{b, min\{-a, w_{n-k}\}\} = b,$$

for $n \geq 0$.

Case 2. Suppose that $a < 0$ and $b < 0$, or $0 < a < -b$, or $0 < b < -a$. From (12), we get

$$z_{n+k+2} = max\{a, min\{-b, z_{n-m}\}\} \in \{a, -b, z_{n-m}\},$$
$$w_{n+m+2} = max\{b, min\{-a, w_{n-k}\}\} \in \{-a, b, w_{n-k}\},$$
$$z_{n+k+3} = max\{a, min\{-b, z_{n-m+1}\}\} \in \{a, -b, z_{n-m+1}\},$$
$$w_{n+m+3} = max\{b, min\{-a, w_{n-k+1}\}\} \in \{-a, b, w_{n-k+1}\},$$
$$\vdots$$
$$z_{n+k+m+2} = max\{a, min\{-b, z_n\}\} \in \{a, -b, z_n\},$$
$$w_{n+k+m+2} = max\{b, min\{-a, w_n\}\} \in \{-a, b, w_n\},$$
$$z_{n+k+m+3} = max\{a, -w_{n+m+2}\} = -w_{n+m+2},$$
$$w_{n+m+k+3} = max\{b, -z_{n+k+2}\}\} = -z_{n+k+2},$$
$$\vdots$$
$$z_{n+2k+m+3} = max\{a, -w_{n+m+k+2}\} = -w_{n+m+k+2},$$
$$w_{n+2m+k+3} = max\{b, -z_{n+k+m+2}\}\} = -z_{n+k+m+2},$$
$$z_{n+2k+m+4} = max\{a, -w_{n+m+k+3}\} = z_{n+k+2},$$
$$w_{n+2m+k+4} = max\{b, -z_{n+k+m+3}\}\} = w_{n+m+2},$$

for $n \geq 0$. If $z_{n+k+2} = z_{n+k+3} = \cdots = z_{n+k+m+2} = -w_{n+m+2} = -w_{n+m+3} = \cdots = -w_{n+m+k+2}$, every solution of (12) is eventually constant. Otherwise, every solution of (12) is eventually periodic with period (*k+m+2*). So, the proof is finished. ■

4. Conclusion

This paper show that every positive solution of the system

$$x_{n+1} = max\left\{A, \frac{C}{y_{n-k}}\right\}, y_{n+1} = max\left\{B, \frac{C}{x_{n-m}}\right\}, n = 0,1,2, \ldots,$$

is bounded and eventually constsnt or eventually periodic with period *k+m+2*. Also, we expose solutions of some special form of this system are bounded and eventually constant or eventually periodic.

References

[1] M. Bayram, E. Daş, On the Positive Solutions of the Difference Equation System $x_{n+1} = 1/y_{n-k}$, $y_{n+1} = x_{n-k}/y_{n-k}$, *Applied Mathematical Sciences,* 4 (2010), 817-821.

[2] D. Clark, M. R. S. Kulenovic, A coupled system of rational difference equation, *Computers and Mathematics with Applications,* 43 (2002), 849-867.

[3] C. Çinar, On the positive solutions of the difference equation system $x_{n+1} = 1/y_n, y_{n+1} = y_n/(x_n x_{n-1})$, *Applied Mathematics and Computation,* 158 (2004), 303-305.

[4] N. Fotiades, G. Papaschinopoulos, On a system of difference equations with maximum, *Applied Mathematics and Computation*, 221 (2013), 684-690.

[5] E. A. Grove, G. Ladas, L. C. McGrath, C. T. Teixeira, Existence and behavior of solutions of a rational system, *Communications on Applied Nonlinear Analysis,* 8 (2001), 1-25.

[6] A. S. Kurbanlı, C. Çinar, M. E. Erdoğan, On the behavior of solutions of rational difference equations $x_{n+1} = x_{n-1}/(y_n x_{n-1} - 1), y_{n+1} = y_{n-1}/(x_n y_{n-1} - 1), z_{n+1} = x\_\{n\}/(y_n z_{n-1})$, *Applied Mathematics*, 2 (2011), 1031-1038.

[7] A. S. Kurbanlı, C. Çinar, D. Simsek, On the Periodicity of Solutions of the System of Rational difference Equations $x_{n+1} = (x_{n-1} + y_n)/(y_n x_{n-1} - 1), y_{n+1} = (y_{n-1} + x_n)/(x_n y_{n-1} - 1)$, *Applied Mathematics*, 2 (2011), 410-413.

[8] B. Ogul, D. Simsek, System Solutions of Difference Equations $x_{n+1} = max\{1/x_{n-4}, y_{n-4}/x_{n-4}\}, y_{n+1} = max\{\frac{1}{y_{n-4}}, \frac{x_{n-4}}{y_{n-4}}\}$, *Kyrgyz State Tech. Uni. I. Razzakov Theo. and App. Sci. Tech. J.*, 34-1, (2015), 202-205.

[9] A. Y. Ozban, On the positive solutions of the system of rational difference equations $x_{n+1} = 1/y_{n-k}$, $y_{n+1} = y_n/(x_{n-m} y_{n-m-k})$, *Journal of Mathematical Analysis and Applications*, 323 (2006), 26-32.

[10] O. Ozkan, A. S. Kurbanlı, On a system of difference equations, *Discrete Dynamics in Nature and Society*, 2013 (2013), 7 pages.

[11] G. Papaschinopoulos, C. J. Schinas, Persistence, oscilatory behavior, and periodicity of the solutions of a system of two nonlinear difference equations, *Journal of Difference Equations and Applications*, 4 (1998), 315-323.

[12] C. J. Schinas, Invariants for difference equations and systems of difference equations of rational form, *Journal of Mathematical Analysis and Applications*, 216 (197), 164-179.

[13] D. Şimşek, B. Demir, C. Çinar, On the solutions of the system of difference equations $x_{n+1} = max\{A/x_n, y_n/x_n\}, y_{n+1} = max\{A/y_n, x_n/y_n\}$, *Discrete Dynamics in Nature and Society*, 2009 (2009), 11 pages. 29 (2001).

[14] D. Şimşek, C. Çinar, I. Yalcinkaya, On the solutions of the difference equations $x_{n+1} = max\{1/x_{n-1}, x_{n-1}\}$, *Int. J. Contemp. Math. Sci.,* 1-10, (2006), 481-487.

[15] I. Yalçınkaya, C. Çinar, D. Şimşek, Global asymptotic stability of a system of difference equation, *Applicable Analysis*, 87 (2008) 677-687.