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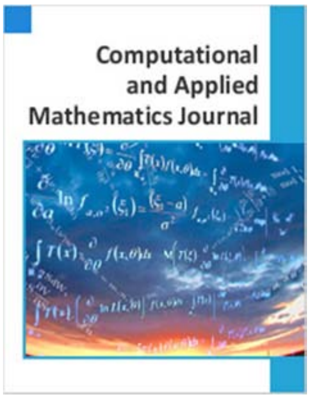

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Numerical Solution of Stochastic Model with Risk Measures via Finite Element Method

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Abstract
In this paper we investigate the use of finite element method (FEM) as a numerical solution of the partial differential equations arising in finance. First the Black-Scholes equation with transaction cost measure and Portfolio risk measure is established. The FEM is then used to transform the differential equation into an algebraic system of equations and to discretize the continuous domain of the problem by means series of simple geometric forms called finite elements, for which the governing relations on the entire continuous domain are valid on each element. Under some assumptions, the approximate solution in the entire continuous domain of the problem is obtained by means of trial function also called the form functions.

1. Introduction

We considered numerical solution of the general Black-Scholes partial differential equation with volatile portfolio risk measure. Black-Scholes [1] obtained option pricingmodel with a constant volatility. It is well known that this model is not consistent with observed option prices. One possible remedy for this is to make the volatility a function of time and the strike price. The price of an option based on option pricing model can be obtained as the solution of a parabolic differential equation.

It is recognized that the options market can help market completeness by providing informational efficiencies. The Black-Scholes model is a well-known model use to price options. The Black-Scholes formula essentially tells investors what value to put on a financial derivative, such as a call option on a stock. Option pricing problems in the investment project evaluation have been solved by the simulation-based methods [2-4] and by the FDM method in [5], delaying the study and the application of other numerical methods like the finite elements method (FEM), which is widely documented and used in others fields of scienceand engineering for decades. There is not a wide literature about the use of the FEM in real option pricing problems, in most cases the numerical solutions to the pde governing the option pricing models (based on the Black & Scholesmodel) are found using the FDM. At the present state of art, some works are related in some way to the FEM applied tothe options pricing problems like the work of [6], who realize a variational analysis of the Back -Scholes equation considering stochastic volatility. Ern et al, [7] used the adaptive FEM to the valuation of European optionswith local volatility, focusing on adaptive control of errors. Zhang [8] studied the American options valuation through an adaptive FEM using a variational formulation. Topper [9], on the other hand, in his technical note, studied in a generalized way the real options pricing using finite element based on a residual formulation.

In this work we investigate the numerical solution of Black Scholes equation with volatile portfolio risk measure by using the finite element method.

2. Stochastic Volatility Model

In order to determine the value of an economic asset, it is needed to take into account at least two aspects of random variability: the growth of the asset and its price. The price of the underlying asset is assumed to be a geometric Brownian motion

$$dS_t = S_t(\mu dt + \sigma dW_t), \qquad (1)$$

and the volatility$\sigma$is allowed to depend on S_t *and* t. Where s_t is the stockprice and t is the time. If the volatility function satisfies suitable regularity conditions, the Black Scholes formula gives the option's price at time $< T$:

$$P_t = \mathbb{E}^*\left(e^{-r((T-t)}(K - S_T)^+ \middle| \mathcal{F}_t\right), \qquad (2)$$

Where$\mathbb{E}^*(|\mathcal{F}_t)$ stands for the conditional expectation with respect to the risk neutral probability.

The Black-Scholes model [10-12] is a powerful tool for valuation of equity options. This model is used for finding prices of stocks. Thismodel considers that the rate of return of the subjacent asset follows a lognormal distribution and its behavior cannot be deterministically determined. To obtain the process followed by the option price, one can show that the Black-Scholes equation takes the form

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial s^2} - rV = 0, \qquad (3)$$

where s is the payoff function,. $S = S(t)$, the stock price,r is the risk-free rate,t is the time since the option was issued,$0 \le t \le T$ and σ is the asset volatility. Equation (3) is a backward moving equation, i.e. it is solved from future to the present time.

To complete the model it is necessary to define appropriate time and boundary conditions associated with (3). Theseconditions depend on the kind of option (call or put; European or American) being evaluated and on the numericalmethod to be applied. A call/put option gives the owner the right to buy/sell the underlying real asset for a certain price ona certain date. The specified price is known as the exercise or strike price and will be denoted by K and the specified date isknown as the expiration date or maturity date and will be denoted by T. An European option can be exercised only at theexpiration date T itself, while an American option can be exercised at any time between the issue date and the expirationdate. On the other hand, the FEM requires defining essential and natural boundary conditions to solve the pde model unlikethe FDM which requires just essential boundary conditions. An essential boundary condition is defined in terms of thevariable's value at the domain's boundary and a natural boundary condition is defined in terms of the variable's derivative value.

Numerical instability may occur when solving (3) numerically due to the fact that it is a PDE with variables coefficients and due to the existence of the convective term [13]. This can be overcome by the following transformation of variables:

Let $s = Ke^x, t = T - \frac{r}{\sigma^2/_2}, andv(x,\tau) = \frac{1}{k}V(s,t)$, Now

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial V}{\partial s}\frac{\partial s}{\partial t} = -\frac{\sigma^2}{2}k\frac{\partial v}{\partial \tau},$$

$$\frac{\partial V}{\partial s} = k\frac{\partial v}{\partial s} = \frac{k}{s}\frac{\partial v}{\partial x},$$

and

$$\frac{\partial^2 V}{\partial s^2} = k\frac{\partial^2 v}{\partial s^2} = \frac{k}{s^2}\left[\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right],$$

Substituting these derivatives in (3) one gets

$$-\frac{\sigma^2}{2}k\frac{\partial v}{\partial \tau} + \frac{\sigma^2}{2}k\left[\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right] + rk\frac{\partial v}{\partial x} - rkv = 0$$

Introducing the measure of the portfolio volatile risk measure, we obtain

$$-\frac{\sigma^2}{2}k\frac{\partial v}{\partial \tau} + \frac{\sigma^2}{2}k\left[\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right] + rk\frac{\partial v}{\partial x} - rkv = (r_{TC} + r_{Vp})x, \quad (4)$$

which is the Black Scholes equation with transaction costs measure and volatile portfolio Risk measure, where r is therisk free rate, σ is the volatility. x is the stock price where$\sigma^2(x,t)$ depends on a solution $u = v(x,t)$ as follows

$$\sigma^2(x,t) = \hat{\sigma}^2(1 - v(x\frac{\partial^2 v}{\partial x^2}(x,t))^{\frac{1}{3}}.$$

$$r_{TC} = \frac{C|\Gamma|\sigma x}{\sqrt{2\Pi}}.\frac{1}{\sqrt{\Delta t}}$$

is the transaction costs measure

$$r_{Vp} = \frac{1}{2}R\sigma^4 x^2\Gamma^2\Delta t$$

is the Volatile portfolio risk measure

$$\Gamma = \frac{\partial^2 v}{\partial x^2}.$$

Minimizing the total risk with respect to the time LagΔtwe have a portfolio with a minimize risk given as

$$-\frac{\sigma^2}{2}k\frac{\partial v}{\partial \tau} + \frac{\sigma^2}{2}k\left[\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right] + rk\frac{\partial v}{\partial x} - rkv = \frac{3}{2}\left(\frac{C^2 R}{2\pi}\right)\sigma^2\left|x\frac{\partial^2 v}{\partial x^2}\right|^{\frac{4}{3}}$$

For simplicity of solution and without loss of generality, we choose the minimized risk as

$$\{min_{\Delta t}(r_{TC}+r_{VP})\}^{\frac{3}{2}} = Ax^2\frac{\partial^2 v}{\partial x^2}, \quad (5a)$$

with

$$A = (\tfrac{3}{2})^{\frac{3}{2}}\left(\frac{C^2R}{2\pi}\right)^{\frac{1}{2}}\hat{\sigma}^3.$$

They change in the value of the portfolio after minimizing the total risk with respect to time lag is given as

$$-\frac{\sigma^2}{2}k\frac{\partial v}{\partial \tau}+\frac{\sigma^2}{2}k\left[\frac{\partial^2 v}{\partial x^2}-\frac{\partial v}{\partial x}\right]+rk\frac{\partial v}{\partial x}-rkv = Ax^2\frac{\partial^2 v}{\partial x^2}, \quad (5b)$$

since transaction cost and risk involved in the business is a drain to the portfolio, we have a change in the portfolio now becomes

$$-\frac{\sigma^2}{2}k\frac{\partial v}{\partial \tau}+\frac{\sigma^2}{2}k\left[\frac{\partial^2 v}{\partial x^2}-\frac{\partial v}{\partial x}\right]+rk\frac{\partial v}{\partial x}-rkv-Ak\left[\frac{\partial^2 v}{\partial x^2}-\frac{\partial v}{\partial x}\right]=0$$

$$\frac{\partial v}{\partial \tau} = \left(1-\frac{A}{\frac{\sigma^2}{2}}\right)\frac{\partial^2 v}{\partial x^2}+\left(\frac{r}{\frac{\sigma^2}{2}}-1+\frac{A}{\frac{\sigma^2}{2}}\right)\frac{\partial v}{\partial x}-\frac{r}{\frac{\sigma^2}{2}}v.$$

Let

$$\beta = \frac{A}{\frac{\sigma^2}{2}}, \theta = \frac{r}{\frac{\sigma^2}{2}}$$

$$\frac{\partial v}{\partial \tau} = (1-\beta)\frac{\partial^2 v}{\partial x^2}+(\theta-1+\beta)\frac{\partial v}{\partial x}-\theta v. \quad (6)$$

Now let

$$\lambda = \frac{1}{2}(\theta-1+\beta), v = \frac{1}{2}(\theta+1+\beta) = \lambda+1$$

So that

$$v^2 = \lambda^2+\frac{1}{4}(\theta+\beta)$$

$$v(x,\tau) = e^{-\lambda x-v^2\tau}u(x,\tau).$$

$$\frac{\partial v}{\partial \tau} = e^{-\lambda x-v^2\tau}(-v^2)u(x,\tau)+\frac{\partial u}{\partial \tau}e^{-\lambda x-v^2\tau}$$

$$= e^{-\lambda x-v^2\tau}\left[-v^2u+\frac{\partial u}{\partial \tau}\right],$$

$$\frac{\partial v}{\partial x} = e^{-\lambda x-v^2\tau}\left[-v^2u+\frac{\partial u}{\partial x}\right],$$

$$\frac{\partial^2 v}{\partial x^2} = e^{-\lambda x-v^2\tau}\left[\lambda^2u-2\lambda\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}\right].$$

Inserting these equation (6) and dividing by $e^{-\lambda x-v^2\tau}$

$$-v^2u+\frac{\partial u}{\partial \tau} = (1-\beta)\left[\lambda^2u-2\lambda\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}\right]+(\theta-1+\beta)\left[-v^2u+\frac{\partial u}{\partial x}\right]-\theta u$$

$$u_\tau = (1-\beta)\frac{\partial^2 u}{\partial x^2}+(-2\lambda+2\beta+\theta-1+\beta)\frac{\partial u}{\partial x}+(\lambda^2-\beta\lambda^2-\theta\lambda\pm\beta\lambda-\theta)u$$

$$u_\tau = (1-\beta)\frac{\partial^2 u}{\partial x^2} \quad (7)$$

Now we have the problem (7) with initial condition $u(x,0) = g(x) = max\{(e^{(\lambda+1)x}-e^{\lambda x}),0\}$, for call option, and $u(x,0) = g(x) = max\{(e^{x}-e^{(\lambda+1)x}),0\}$, for put option.

Applying the Fourier transformation with respect to $x$in equation (7) we have

$$\frac{\partial}{\partial \tau}\hat{u}(\xi,\tau) = (1-\beta)(i\xi)^2\hat{u}(\xi,\tau),$$

where

$$\hat{u}(\xi,\tau) = \int_{-\infty}^{\infty}u(x,\tau)e^{2\pi ix\xi}\,dx.$$

Thus the solution in the Fourier domain can be presented as

$$\hat{u} = (1-\beta)\hat{u}(\xi,0)e^{-\xi^{2\tau}}.$$

Now the initial function is $u(x,0) = g(x)$. Hence applying the inverse Fourier transformation

We write the exact solution as

$$u(x,\tau) = \frac{(1-\beta)}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty}g(\xi)\,e^{-\frac{(x-\xi)^2}{4\tau}}d\xi.$$

3. Brief on Finite Element Method

The finite element method (FEM) is a numerical technique for finding approximate solutions to boundary value problems for partial differential equations. It is also referred to as finite element analysis (FEA). It subdivides a large problem into smaller, simpler parts that are called finite elements. The simple equations that model these finite elements are then assembled into a larger system of equations that models the entire problem. FEM then uses variational methods from the calculus of variations to approximate a solution by minimizing an associated error function. A typical work out of the method involvesdividing the domain of the problem into a collection of subdomains, with each subdomain represented by a set of element equations to the original problem, followed by systematically recombining all sets of element equations into a global system of equations for the final calculation. The global system of equations has known solution techniques, and can be calculated from the initial values of the original problem to obtain a numerical answer.

In the first step above, the element equations are simple equations that locally approximate the original complex equations to be studied, where the original equations are often partial differential equations (PDE). To explain the approximation in this process, FEM is commonly introduced as a special case of Galerkin method. The process, in mathematical language, is to construct an integral of the inner product of the residual and the weight functions and set the integral to zero. In simple terms, it is a procedure that minimizes the error of approximation by fitting trial functions into the PDE. The residual is the error caused by the trial functions, and the weight functions are polynomial approximation functions that project the residual. The process eliminates all the spatial derivatives from the PDE, thus approximating the PDE locally with

- a set of algebraic equations for steady state problems,
- a set of ordinary differential equations for transient problems.

These equation sets are the element equations. They are linear if the underlying PDE is linear, and vice versa. Algebraic equation sets that arise in the steady state problems are solved using numerical linear algebra methods, while ordinary differential equation sets that arise in the transient problems are solved by numerical integration using standard techniques such as Euler's method or the Runge-Kutta method.

In step (2) above, a global system of equations is generated from the element equations through a transformation of coordinates from the subdomains' local nodes to the domain's global nodes. This spatial transformation includes appropriate orientation adjustments as applied in relation to the reference coordinate system. The process is often carried out by FEM software using coordinate data generated from the subdomains.

FEM is best understood from its practical application, known as finite element analysis (FEA). FEA as applied in engineering is a computational tool for performing engineering analysis. It includes the use of mesh generation techniques for dividing a complex problem into small elements, as well as the use of software program coded with FEM algorithm. In applying FEA, the complex problem is usually a physical system with the underlying physics such as the Euler-Bernoulli beam equation, the heat equation, or the Navier-Stokes equations expressed in either PDE or integral equations, while the divided small elements of the complex problem represent different areas in the physical system.

FEA is a good choice for analysing problems over complicated domains (like cars and oil pipelines), when the domain changes (as during a solid state reaction with a moving boundary), when the desired precision varies over the entire domain, or when the solution lacks smoothness. For instance, in a frontal crash simulation it is possible to increase prediction accuracy in "important" areas like the front of the car and reduce it in its rear (thus reducing cost of the simulation). Another example would be in numerical weather prediction, where it is more important to have accurate predictions over developing highly nonlinear phenomena (such as tropical cyclones in the atmosphere, or eddies in the ocean) rather than relatively calm areas.

4. Discretization of Variables in the FEM

The Finite Element method (FEM) is an approximate method that allows solving the PDE model governing the option pricing problems. The FEM discretizes the continuous domain of the problem by means series of simple geometric forms called finite elements, for which the governing relations on the entire continuous domain are valid on each element. Under this assumption, the approximate solution in the entire continuous domain of the problem can be obtained by means of trial function also called the form functions. The FEM transforms the differential equation into an algebraic system of equations which can then be solved easily by known numerical methods.

Consider now, the case of valuing an European call option $V(S,t)$ through the diffusion forward moving Black-Scholes model defined previously as in (7). To solve this problem through the FEM it is necessary to discretize appropriately the time –space domain of the problem,

$$\Omega_{\tau,x} = \left\{(\tau,x)|0 \le \tau \le \frac{\sigma^2 T}{2}, x_{-\infty} \le x \le x_{+\infty}\right\}.$$

For this purpose, the Kantorovitch's discretization is used as in [13]. This kind of discretization element is a partial discretization by finite the time domain Ω_τ is discretized by finite difference, see [14,15]. Using Kantorovitch's discretization the approximate solution is obtained at each time τ for the space domain $\Omega_{\tau,x}$:

$$\tau_i = t_0 + i\Delta\tau, i = 0,1,\dots n, \tag{8}$$

$$x_i = x_{-\infty} + (j-1)\Delta x, j = 1,2,\dots,m+1. \tag{9}$$

Using the above discretization we obtain a regular mesh of $(n+1)(m+1)$ of discrete points (τ_i, x_j) for independent variables of the model. In this way, the domain's intervals $[0, \sigma^2 T/2]$ and $[x_{-\infty}, x_\infty]$ are divided in constant length subintervals $\Delta\tau$ and Δx defined by

$$\Delta\tau = \frac{\tau_n - \tau_0}{n} = \frac{\delta^2 T}{2n},$$

$$\Delta x = \frac{x_{m+1} - x_1}{m} = \frac{x_\infty - x_{-\infty}}{m}.$$

The finite element can be defined by two adjacent discrete points $x_j\ and\ x_{j+1}$ and or by three adjacent discrete points$x_j, x_{j+1}\ and\ x_{j+2}$. The number of finite elements in $\Omega_x\ is\ m\ and\ m/2$ respectively in the first and the second case. For $m+1>3, m$ is an even number.ξdenotes the local coordinate which can take integer values only. It is used to define all variables at each finite element. At each j finite element it is necessary to define the approximate solution v and the space variable x as a combination of its values at the finite element's nodes. For this purpose the form functions $N_k(\xi)$, which is defined in terms of the local coordinate, are used. The Lagrangian form functions are commonly used. These are linear or quadratic type depending on the required approximation. The linear form functions are required for thefinite element with two nodes, and the quadratic function is needed for the elements with three nodes; see [13]. It can be seen that each form function $N_j(\xi)$, is related to a finite element's node j [14],which takes the value of unity at j and zero at all other nodes. The approximate solution v and the space variable x, whenrestricted to a typical finite element can be written as a linear combination of node values using Lagrangian linear formfunctions as:

$$v = \sum_{k=j}^{jk} N_k(\xi) q_k(\tau) = \frac{1-\xi}{2} q_j + \frac{1+\xi}{2} q_{j+1}, \tag{10}$$

$$x = \sum_{k=j}^{jk} N_k(\xi) x_k = \frac{1-\xi}{2} x_j + \frac{1+\xi}{2} x_{j+1}, \tag{11}$$

where jk denotes the number of nodes in the finite element. The solution v and the space variable x can be written as a quadratic combination of node values using Lagrangian quadratic form functions as

$$v = \sum_{k=j}^{jk} N_k(\xi) q_k(\tau) = -\xi \frac{1-\xi}{2} q_j + (1-\xi)(1+\xi) q_{j+1} + \xi \frac{1+\xi}{2} q_{j+2}, \tag{12}$$

$$x = \sum_{k=j}^{jk} N_k(\xi) x_k = -\xi \frac{1-\xi}{2} x_j + (1-\xi)(1+\xi) x_{j+1} + \xi \frac{1+\xi}{2} x_{j+2} \tag{13}$$

when using the same form functions to define the approximate solution v and the space variable x an isoparametric formulation is obtained from which the values q_j need to be found to know the approximate solution of the option pricing problem.

5. Integral Formulation of the Partial Differential Equation

Once the domain has been discretized it is necessary to find the value of time dependent parameters $q_j(\tau)$, called thegeneralized variables, which allow one to write the approximate solution at each finite element using equations (10) or (12).

To get these generalized variables, it is necessary to define the weak or integral formulation of the differential equation ofthe option pricing problem which is obtained by the Galerkin weighted residue method.

The Galerkin weighted residue method obtains the generalized variables $q_j(\tau)$, by minimizing the residue, say R. Theresidue, R, is the result of the exact solution being replaced by the approximate solution in $L(v) - f = 0$, where L is thedifferential operator and f a function of independent variables. Hence, $= L(v) - f$, where v is an approximate solution.

The residue R is minimized to zero by weighting it with the so-called weight functions wk. In the Galerkin method, the form functions are used as the weight functions e.g., $w_k = N_k(\xi)$. This results in the equation

$$\int_{\Omega} w_k R d\Omega = \int_{\Omega} w_k (L(v) - f) d\Omega = \int_{\Omega} N_k(\xi) \left((1-\beta) \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial \tau} \right) d\Omega = 0 \tag{14}$$

to be solved,where k represents all nodes in Ω_x. Using $N^T = \left(N_j(\xi), N_{j+1}(\xi), \dots N_k(\xi)\right)$,the approximate solution $v = N.q$, $x = N.X, X^T = \left(x_j. x_{j+1}, \dots, x_k\right)$ and $q^T = \left(q_j, q_{j+1,} \dots, q_k\right)$, the expression (14) can be written as

$$\int_{x_j}^{x_k} (1-\beta) N^T \frac{\partial^2 (N.q)}{\partial x^2} dx - \int_{x_j}^{x_k} N^T \frac{\partial^2 (N.q)}{\partial \tau} dx = 0. \tag{15}$$

Changing to local variables for each element and integrating by part the first integral of (15) one has

$$(1-\beta) N^T \frac{\partial N}{\partial x} q \Big|_{-1}^{1} - (1-\beta) \int_{-1}^{1} \frac{\partial N^T}{\partial x} \frac{\partial N}{\partial x} q J d\xi - \int_{-1}^{1} N^T N \frac{\partial q}{\partial \tau} J d\xi = 0. \tag{16}$$

where $J = \frac{\partial x}{\partial \xi}$. It can be shown using (11) and (13) respectively, that $J\, equals\ {}^{\Delta x}/_{2}$ for the two node finite element Δx for the three node finite element. The term $N^T \frac{\partial N}{\partial x} q \Big|_{-1}^{1}$ represents the natural boundary conditions evaluated at the extreme nodes of the element and defined by the column arrays

$$F^e = \left[-\frac{\partial v_j}{\partial x}, \frac{\partial v_{j+1}}{\partial x} \right]^T \tag{17}$$

$$F^e = \left[-\frac{\partial v_j}{\partial x}, 0, \frac{\partial v_{j+1}}{\partial x} \right]^T \tag{18}$$

for two and three node finite elements respectively.

Equation (16) demands that the form function used be a C_0 class function. This means that (16) requires a continuous form function in the domain; this restriction is accomplished by the Lagrangian form function used in this work. Rearranging and introducing the array of form function's derivatives

$$B = \frac{\partial N}{\partial x} = \frac{\partial N}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial N}{\partial \xi} = \frac{\partial N}{\partial \xi} J^{-1} = \left(\frac{\partial N_j}{\partial \xi}, \frac{\partial N_{j+1}}{\partial \xi}, \dots, \frac{\partial N_k}{\partial \xi} \right) J^{-1},$$

so that (16) can be rewritten for each finite element as follows:

$$\left(\int_{-1}^{1} N^T N J d\xi \right) \dot{q} + (1-\beta) \left(\int_{-1}^{1} B^T B J d\xi \right) q = F^e, \tag{19}$$

which is an algebraic system. Equation (19) can be re-written as

$$C^e\dot{q} + (1-\beta)K^e q = F^e, \tag{20}$$

where $\dot{q}$ denotes the temporal derivative of the generalized variable,$\frac{\partial q}{\partial \tau}$ and

$$C^e = \left(\int_{-1}^{1} N^T NJd\xi\right) \text{ and } K^e = \left(\int_{-1}^{1} B^T BJd\xi\right).$$

The system of equations presented by (20) is valid just for a single finite element. Therefore, it is necessary to find the general system of equations that defines the approximate solution in the entire domain Ω at a certain time τ. The system of equations for the entire finite system is represented by

$$C\dot{q} + (1-\beta)Kq = F, \tag{21}$$

where $k = \sum_e K^e, C = \sum_e C^e andF = \sum_e F^e$, the sum is taken over all finite elements.

Discretization of time variable

To solve the algebraic system (21) it is necessary to approximate the temporal derivative of the generalized variable $\partial q/\partial\tau$ at τ_i. For this purpose, a backward finite difference approximation is used, ie.$\dot{q}\ at\ \tau_i$ is approximated by

$$\dot{q}^i = \frac{q^i - q^{i-1}}{\Delta\tau}, \tag{22}$$

Where $\Delta\tau$=$\tau^i - \tau^{i-1}$. Substituting (22) and (21) we get

$$\left[\frac{1}{\Delta\tau}C + K\right]q^i = \left[\frac{1}{\Delta\tau}C\right]q^{i+1} + F, i = 1,2,\dots,n$$

Where n is the number of nodes in the time domain Ω_t. The above system of equations is of the form $Aq = b$, which can be solved to obtain q^i at τ_i in terms of its previous value $q^{i-1}\ at\ \tau_{i-1}$.

6. The A Posteriori Estimate

If we consider a partition of the interval $[0,T]$ into subintervals $[t_{m-1}, t_m], 1 \le m \le N$, such that

$$0 = t_0 < t_1 \dots < t_N = T.$$

Set $\Delta t_m := t_m - t_{m-1}, \Delta t := max\{\Delta t_n | 1 \le m \le N\}$ and

$$\rho\Delta t_n := \max_{2\le m\le N} \frac{\Delta t_m}{\Delta t_{m-1}}. \tag{23}$$

For continuous of f on $[0,T]$, we introduce the notation $f^m = f(t_m)$.

The semi discrete problem arising from the implicit Euler Scheme is as follows: Find $(u^m)_{0\le m\le N} \in V$ such that

$$(u^m - u^{m-1}. v)_{0,\Omega} + \Delta t_m a_{t_m}(u^m, v) = 0. v \in V, 1 \le m \le N \tag{24a}$$

$$u^0 = u_0. \tag{24b}$$

The existence and uniqueness of solution $u^m \in V$ of (24a), (24b) can be shown for sufficiently small timestep Δt_m.

Theorem 1: Under the following assumption: A) The function σ is continuously differentiable, and there exist constants $0 < \sigma_{min} \le \sigma_{max}$ and $C_\sigma > 0$, such that for all $(x,t) \in \Omega \times [0,T]$ there holds

$$\sigma_{min} \le \sigma(x,t) \le \sigma_{max}, \tag{25}$$

$$\left|x\frac{\partial\sigma}{\partial x}(x,t)\right| \le C_{\sigma,}$$

B) The function r is continuous and nonnegative on $[0,T]$ and the time step-restriction

$$\Delta t_m < \frac{1}{2\lambda}.$$

The semi discrete problem (24a) and (24b) admits a unique solution (see prove [13]).

Given a null sequence $\mathcal{H}$ of positive real numbers, for the discretization of the semi discrete problem (24a,) and (24b) in space, we use continuous, piecewise linear finite elements with respect to a family of simplicial triangulations

$$\mathcal{T}_{mh}, 1 \le m \le N, of\ \Omega\ forT \in \mathcal{T}_{mh},$$

we denote by $x_{min}(T). x_{rmax}(T)$ the endpoints of T and refer to $h_T := x_{max}(T) - x(T)_{min}$ as the length of T and to $h_m := max\{h_T | T \in \mathcal{T}_{mh}\}$ as maximal size of the interval in $\mathcal{T}_{mh}$. Moreover. for $D \subseteq \Omega$ we refer to $N_{mh}(D)$ as the set of nodes of $\mathcal{T}_{mh}\ in\ D$ and associate with each $T \in \mathcal{T}_{mh}$the patch w_T according to

$$w_T := \bigcup\{T' \in \mathcal{T}_{mh} | N_{mh}(T') \cap N_{mh}(T) \neq \emptyset\}$$

We assume that the family of triangulations is locally quasi-uniform in the sense that there exists a constant $\rho > 0$ such that for two adjacent elements $T, T' \in \mathcal{T}_{mh}$there holds

$$h_T \le ph_{T'}, h \in \mathcal{H}$$

For each $h \in \mathcal{H}$, we define the finite element spaces by

$$V_{nh} := \{v_h^m \in C^0(\Omega) | v_h^m|_T \in P^1(T), T \in \mathcal{T}_{mh}\},$$

$$V_{mh}^0 := V_{mh} \cap V_0$$

Where $P^1(T)$ stands for the linear space of polynomials of degree 1 $on\ T$.

Assuming that $u_0 \in V_{1h}$. The fully discrete problem reads as follows: Find $(u_h^m)_{1\le m\le N}, u_h^m \in V_{mh}^0, 1 \le m \le N$, such that

$$\left(u_h^m - u_h^{m-1}, v_h\right)_{0,\Omega} + \Delta t_m a_{t_m}(u_h^m, v_h) = 0. v_h \in V_{mh}^0, \tag{26a}$$

$$u_h^0 = u_0, \tag{26b}$$

If theorem one holds true. Then, the fully discrete problem admits a unique solution.

For the fully discretized Black-Scholes equations (26a) and (26b), the global discretization error $u - u_{h,\delta t}$ can be assessed by a time error estimator and a price error estimator. The time error estimator is local in time and global in price. It is given by

$$\eta_m := \sqrt{\delta t_m} e^{-\lambda t_{m-1}} \frac{\sigma_{min}}{\sqrt{2}} |u_h^m - u_h^{m-1}|_V, 1 \le m \le N, \tag{27}$$

where $\sigma_{min} > 0\ and\ \lambda \geq 0$ are constants from the assumption(23) and Garding'sine quality. On the other hand, the price error estimator is local both in time and price. It is given by

$$\eta_{mT} := \frac{h_T}{x_{max(T)}} \left\| R_T\left(u_h^{m-1}.u_h^m\right) \right\|_{0,T}. T \in T_{mh}. 1 \leq n \leq N, \quad (28)$$

Where $R_T(u_h^{m-1}.u_h^m)$ stands for residual with respect to the strong form (6) of the Adjusted Black Scholes equation

$$R_T(u_h^{m-1}.u_h^m) := \frac{u_h^m - u_h^{m-1}}{\delta t_m} - \frac{\sigma^2 x^2}{2}\frac{\partial^2 u_h^m}{\partial x^2} - rx\frac{\partial u_h^m}{\partial x} + ru_h^m + Ax^2\frac{\partial^2 u_h^m}{\partial x^2}$$

Moreover, let η_m and $\eta_{m,w}$ be the time error and price error estimators given by (27) and (28), respectively. Then there exist a positive constant $\alpha \leq \frac{1}{2}$ such that for $\lambda \delta t \leq \alpha$ there holds

$$\left[\left[u - u_{h,\delta t}\right]\right](t_m) \lesssim \left(\frac{C}{\sigma_{min}^2} C(u_0 \delta t) + \frac{\mu}{\sigma_{min}^2} \left(\sum_{m-1}^{n} \eta_m^2 + \frac{\delta t_m}{\sigma_{min}^2} \kappa(\rho \delta t) \prod_{i=1}^{m-1} (1 - 2\lambda \delta t_i) \sum_{T \in T_{mh}} \eta_{m,T}^2 \right)^{1/2} \right).$$

where $C; = 4C_1 + 2C_2 + C_3.C(u_0)$ is given by

$$\left(max(2{,}1 + \rho\delta t)\|u_0\|^2 + \frac{1}{2}\sigma_{min}^2 \delta t_1 |u_0|_V^2 \right)^{\frac{1}{2}}.$$

and

$$\kappa(\rho \delta t) := (1 + \rho \delta t)^2 \|u_0\|^2 + \max(2{,}1 + \rho \delta t),$$

7. Conclusion

Solution to the adjusted Black-Scholes equation by the finite element method was discussed where the exact solution was first obtained by applying the inverse Fourier transform. Then finite element method, because it allows for a more rigorous analysis of problems with discontinuous data and posteriori error estimates is applied. With the FEM the differential equation was transform into an algebraic system of equations which can then be solved easily by known numerical methods. There after the price error estimator in time and price was determined.

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