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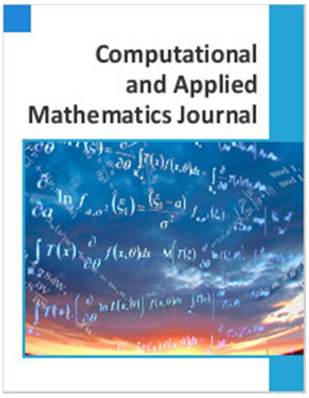

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On a Reliable Technique for Solving Parabolic Differential Equation

Adnan Rashid[1], Muhammad Ashraf[1], Qazi Mahmood Ul-Hassan[1], Kamran Ayub[2], Muhammad Yaqub Khan[2]

[1]Department of Mathematics, Faculty of Basic Sciences, University of Wah, Wah Cantt., Pakistan
[2]Department of Mathematics, Riphah International University, Islamabad, Pakistan

Email address
qazimahmood@yahoo.com (Q. M. Ul-Hassan)

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Abstract
Solutions of nonlinear models are of great importance and their significance has increased a lot. In given paper, the homotopy perturbation method (HPM) is implemented to solve the linear and nonlinear parabolic equations with proper initial conditions. The resemblance of the analytical solutions attained by HPM with exact solution allows the order of this method. The results show accuracy and efficiency of HPM in solving the parabolic equation.

1. Introduction

Many physics and engineering problems are modeled by partial differential equations. In many instances these equations are nonlinear and exact solutions are very difficult to obtain. Numerical methods were developed in order to find approximate solutions to these nonlinear equations. However, numerical solutions are insufficient to determine general properties of certain systems of equations and thus analytical methods have been developed.

In this study we discuss solutions to linear and nonlinear parabolic equations by analytic technique. The aim is to compare results of analytical method and focusing on accuracy, convergence and computational efficiency. The homotopy perturbation was first planned by chinese mathematcian Ji-Huan He. İnstead classical techniques, the homotopy perturbation method deals to an analytical approximate and exact solutions of the nonlinear equations easily without generating the equation or linearizing the problem with high correctness and less calculation and not used comples assumtions. İt (HPM) is used as analytical tool to solve mathematical models without any calculation in a given equation. İt is not effected by computation round-off errors and is not using large memory and time. Through this technique we simplify large number of nonlinear problems. Through this study, we used homotopy perturbation method to simplify the parabolic equations. The numerical results are compared with exact solutions. The errors are very less in these problems. Now a days Javidi and Golbabai used Adomian decomposition method (ADM) for approximating the solution of parabolic equations.

In addition there is no general procedure which is applicable for all such equations. So each equation has to be studied by considering as an individual problem. For this goal, many novel methods for the detection of exact travelling wave solutions of NLEEs have been drawn in huge combinations by a large number of experts. As a result, a lot of work has been done in formulation of several convincing and significant techniques. Different researchers apply these techniques on mathematical and physical models. Such as, the homogenous balance method (Wang, 1995; Zayed, Zedan and Gepreel, 2004), Hirota’s bilinear transformation method (Hirota, 1973; Hirota and Satsuma, 1981), auxiliary equation method (Sirendaoreji, 2004), trial function method (Inc and Evans, 2004), Jacobi elliptic function method (Ali, 2011), tanh-function method (Abdou, 2007; Fan, 2000; Malfliet, 1992), homotopy perturbation method (Mohyud-Din et al., 2011), sine-cosine method (Wazwaz, 2004; Bibi and S. T. Mohyud-Din, 2013), truncated Painleve expansion method (Weiss et al., 1983), variational iteration method (He, 1997; Abdou and Soliman, 2005; Abbasbandy, 2007), Exp-function method (He and Wu, 2006; Akbar and Ali, 2012; Naher et al., 2012), (G'/G)-expansion method (Wang et al., 2008; Akbar et al., 2012; Zayed, 2010; Zayed and Gepreel, 2009; Ali, 2011; Zayed, 2009; Shehata, 2010), improved (G'/G)-expansion method (Zhang, F et al., 2010), exp(-φ(ξ))-expansion method (Khan and Akbar, 2013) and so on.

Recently many methods were suggested to search for traveling wave solutions for various nonlinear wave equations, e.g., Backlund transformation [1, 2], Hirota's bilinear method [3], inverse scattering method [4], extended tanh method [5–7], Adomian Pade approximation [8–10], variational method [11–14], the variational iteration methods [15, 16], various Lindstedt–Poincare methods [17–20], and others [21–28]. This paper applies the homotopy perturbation method [29–31] to the discussed problem.

2. Homotopy Perturbation Method

For a general nonlinear boundary-conditioned differential equation

$$A(u)-f(r)=0, r\in\Omega \tag{1}$$

Where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, u is the r-dependent unknown function is the boundary of the domain Ω.

The operator A can, generally speaking, be divided in to two parts L and N, where L is linear, and N is nonlinear, therefore

Eq. (1) can be written as

$$L(u)+N(u)-f(r)=0 \tag{2}$$

By using the homotopy technique, one can construct a homotopy $V(r,p):\Omega\times[0,1]\to A$ which satisfies

$$H(v,p)=(1-p)\left[L(v)-L(u_0)\right]+p\left[A(v)-f(r)\right]=0 \tag{3}$$

Or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p\left[A(v)-f(r)\right]=0 \tag{4}$$

Where $p\in[0,1]$ is an embedding parameter, and u_0 is the initial approximation which satisfies the boundary conditions

Clearly, we have

$$H(v,0)=L(v)-L(u_0)=0 \tag{5}$$

And, we also have

$$H(v,1)=A(v)-f(r)=0 \tag{6}$$

The changing process of p from zero to unity is just that of $V(r,p)$ changing from $u_0(r)$ to $u(r)$. This is called deformation, and also, $L(v)-L(u_0)$ and $A(v)-f(r)$ are called homotopic in topology. If the embedding parameter p; $(0\le p\le 1)$ is considered as a ''small parameter'', applying the classical perturbation technique, we can assume that the solution of Eq. (3) can be given as a power series in p, i.e.

$$v=v_0+pv_1+p^2v_2+\ldots \tag{7}$$

and after setting p = 1 results in the approximate solution of Eq. (1) as

$$u=\lim_{p\to 1} v=v_0+v_1+v_2+\ldots$$

3. Solution Procedure

In this paper, homotopy perturbation method has been applied on linear and nonlinear parabolic equations with initial condition, namely we consider

$$\frac{dw}{dt}=\frac{d^2w}{dx^2}+\psi(u)+h(x,t),\ (x,t)\in[a,b]\times(0,T), \tag{8}$$

with the initial condition

$$w(x,0)=p(x) \tag{9}$$

where Ψ is a function of *w*. It is search for the solution satisfying Eqs. (8) - (9).

Problem 1

This problem was used by Hopkins and Wait [24] to provide an example of a problem with a nonlinear source term

$$\frac{dw}{dt}=\frac{d^2w}{dt^2}+e^{-w}+e^{-2w},\ (x,t)\in[0,1]\times(0,1]. \tag{10}$$

with the initial condition *w(x,0) = ln(x+2)*.

We have

$$\Psi(w)=e^{w}+e^{-2w},\ h(x,t)=0 \text{ and } p(x)=\ln(x+2).$$

The constructed homotopy is given below

$$\frac{dw}{dt}-\frac{dw_0}{dt}=v\left(\frac{d^2w}{dx^2}+e^{-w}+e^{-2w}-\frac{dw_0}{dt}\right), \tag{11}$$

Assume the solution of Eq. (11) to be in the form

$$w = w_0 + v w_1 + v^2 w_2 + v^3 w_3 + ... \quad (12)$$

Substituting (12) into (11) and equating the coefficients of like powers v, the following set of differential equations have been obtained

$$v^0 : \frac{dw_0}{dt} - \frac{dw_0}{dt} = 0$$

$$v^1 : \frac{dw_1}{dt} = \frac{d^2 w_0}{dx^2} + e^{-w_0} + e^{-2w_0} - \frac{dw_0}{dt}$$

$$v^2 : \frac{dw_2}{dt} = \frac{d^2 w_1}{dx^2} + w_1(-e^{-w_0} - 2e^{-2w_0})$$

$$v^3 : \frac{dw_3}{dt} = \frac{d^2 w_2}{dx^2} + (-w_2 + \frac{1}{2} w_1^2) e^{-w_0} + (-2w_2 + 2w_1^2) - \frac{1}{48} w_2 w_1^2) e^{-2w_0}$$

$$\vdots$$

solving the above equations, we obtain

$$w_0 = \ln(x+2),$$

$$w_1 = \frac{t}{x+2},$$

$$w_2 = \frac{-t^2}{2(x+2)^2},$$

$$w_3 = \frac{t^3}{3(x+2)^3},$$

$$\vdots$$

$$w_n = \frac{(-1)^{m+1} t^m}{m(x+2)^m},$$

and so on. Therefore from the results we have

$$w(x,t) = ln(x+2) + \frac{t}{x+2} - \frac{t^2}{2(x+2)^2} + \frac{t^3}{3(x+2)^3} + ... + \frac{(-1)^{m+1} t^m}{m(x+2)^m} + ...$$

$$= ln(x+2) + \ln(\frac{t}{x+2} + 1)$$

$$= ln(x + t + 2)$$

which is the exact solution of the problem

Problem 2

This problem was used by Lawson and et. al. [25] as the form:

$$\frac{dw}{dt} = \frac{d^2 w}{dx^2} + (\pi^2 - 1 - v)w + (ve^{-t} + e^{-vt}) \; (x,t) \in [0,1] \times (0,1] \quad (13)$$

with the initial condition

$$w(x,0) = 2\sin(\pi x) \quad (14)$$

Here we have $\Psi(w) = (\pi^2 - 1 - v)w$, $h(x,t) = ve^{-t} + e^{-vt}$ and $p(x) = 2\sin(\pi x)$.

The constructed homotopy is given as

$$\frac{dw}{dt} - \frac{dw_0}{dt} = v(\frac{d^2 w}{dx^2} + (\pi^2 - 1 - v)w + (ve^{-t} + e^{-vt}) - \frac{dw_0}{dt}), \quad (15)$$

Substituting (12) into (15) and equating the coefficients of like powers v, the following set of differantial equations have been obtained

$$v^0 : \frac{dw_0}{dt} - \frac{dw_0}{dt} = 0$$

$$v^1 : \frac{dw_1}{dt} = \frac{d^2 w_0}{dx^2} + (\pi^2 - 1 - v)w_0 + (ve^{-t} + e^{-vt}) - \frac{dw_0}{dt}$$

$$v^2 : \frac{dw_2}{dt} = \frac{d^2 w_1}{dx^2} + (\pi^2 - 1 - v)w_1$$

$$v^3 : \frac{dw_3}{dt} = \frac{d^2 w_2}{dx^2} + (\pi^2 - 1 - v)w_2$$

$$\vdots$$

solving the above equations, we obtain

$$w_0 = [2 - ve^{-t} - \frac{1}{v} e^{-vt} + v + \frac{1}{v}]\sin(\pi x),$$

$$w_1 = -(1+v)[2t + ve^{-t} + \frac{1}{v^2} e^{-vt} + (v + \frac{1}{v})t - (v + \frac{1}{v^2})]\sin(\pi x),$$

$$w_2 = (1+v)^2 [t^2 - ve^{-t} - \frac{1}{v^3} e^{-vt} + (v + \frac{1}{v}) \frac{t^2}{2!} - (v + \frac{1}{v^2})t + (v + \frac{1}{v^3})]\sin(\pi x),$$

and so on. Therefore from the equations, we have

$$w(x,t)=[2-ve^{-t}-\frac{1}{v}e^{-vt}+v+\frac{1}{v}]\sin(\pi x)-(1+v)[2t+ve^{-t}+\frac{1}{v^2}e^{-vt}+(v+\frac{1}{v})t-(v+\frac{1}{v^2})]\sin(\pi x)$$

$$+(1+v)^2[t^2-ve^{-t}-\frac{1}{v^3}e^{-vt}+(v+\frac{1}{v})\frac{t^2}{2!}-(v+\frac{1}{v^2})t+(v+\frac{1}{v^3})]\sin(\pi x)+..........$$

$$w(x,t)=(e^{-t}+e^{-vt})\sin\pi x$$

Table 1. *Absolute error for various values of x, t and s (number of terms) for test problem 1.*

x/t	**0,2**	**0,4**	**0,6**	**0,8**	**1**
s=5					
0,2	8,7288E-8	5,2113E-6	5,5632E-5	2,9414E-4	0,0011
0,4	5,2100E-8	3,1268E-6	3,3532E-5	1,7801E-4	6,4360E-4
0,6	3,2396E-8	1,9530E-6	2,1027E-5	1,1202E-4	4,0630E-4
0,8	2,0859E-8	1,2624E-6	1,3640E-5	7,2890E-5	2,6512E-4
1	1,3842E-8	8,4059E-7	9,1099E-6	4.8819E-5	1,7801E-4
s=10					
0,2	2,9388E-13	5,5942E-10	4,5169E-8	1,0028E-6	1,0989E-5
0,4	1,1369E-13	2,1740E-10	1,7638E-8	3,9323E-7	4,3255E-6
0,6	4,6851E-14	9,1057E-11	7,4176E-9	1,6601E-7	1,8321E-6
0,8	2,1094E-14	4,0656E-11	3,3245E-9	7,4639E-8	8,2616E-7
1	9,9920E-15	1,9182E-11	1,5736E-9	3,5433E-8	3,9323E-7
s=20					
0,2	2,2204E-16	3,3307E-16	5,2625E-14	2,1011E-11	2,1403E-9
0,4	1,1102E-16	6,6613E-16	9,1038E-15	3,4535E-12	3,5318E-10
0,6	4,4409E-16	2,2204E-16	1,5543E-15	6,5525E-13	6,7235E-11
0,8	2,2204E-16	2,2204E-16	2,2204E-16	1,4078E-13	1,4458E-11
1	2,2204E-16	6,6613E-16	6,6613E-16	3,4195E-14	3,4537E-12

Table 2. *Absolute error for various values of x, t, v and s (number of terms) for test problem 2.*

x/t	**0,2**	**0,4**	**0,6**	**0,8**	**1**
s=20,v=1					
0,2	4,9449E-13	3,5554E-12	5,4122E-10	2,3279E-11	2,4691E-10
0,4	8,0003E-13	5,7551E-12	8,7571E-10	3,7667E-11	3,9951E-10
0,6	8,0025E-13	5,7527E-12	8,7571E-10	3,7667E-11	3,9951E-10
0,8	4,9438E-13	3,5553E-12	5,4122E-10	2,3280E-11	2,4691E-10
1	1,0304E-28	7,4077E-28	1,1276E-25	4,8503E-27	1,1444E-26
s=10,v=2					
0,2	1,0967E-10	2,0841E-7	1,7609E-5	4,0750E-4	0,0046
0,4	1,7745E-10	3,3721E-7	2,8462E-5	6,5934E-4	0,0075
0,6	1,7745E-10	3,3721E-7	2,8492E-5	6,5934E-4	0,0075
0,8	1,0967E-10	2,0841E-7	1,7609E-5	4,0750E-4	0,0046
1	2,2850E-26	4,3421E-23	3,6689E-21	8,8901E-21	9,6653E-19
s=20,v=3					
0,2	3,8205E-4	5,1061E-4	6,6805E-5	2,8638E-4	3,1596E-4
0,4	6,1818E-4	8,2618E-4	1,0809E-4	4,6338E-4	5,1124E-4
0,6	6,1817E-4	8,2618E-4	1,0809E-4	4,6338E-4	5,1124E-4
0,8	3,8205E-4	5,1061E-4	6,6805E-5	2,8638E-4	3,1596E-4
1	7,9600E-20	1,0639E-19	1,3919E-20	5,9668E-20	6,5830E-20

4. Conclusion

In the paper, the homotopy perturbation method is applied to solve linear and nonlinear parabolic equations. The advantage of the method is that it does not need a small parameter in the system, leading to wide application in nonlinear differential equations. With the help of some mathematical software, such as MAPLE, MATLAB, the method provides a powerful mathematical tool to more complex nonlinear systems. The obtained results are very encourging. It is concluded that under discussion technique is user friendly with minimum computational work, also we can extend it for physical problems of different nature.

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