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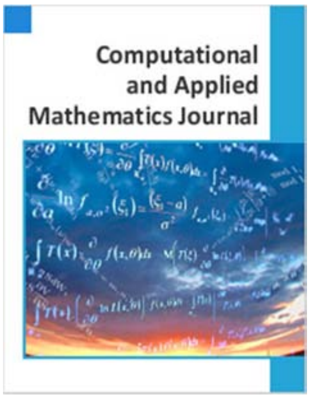

Solving Initial and Boundary Value Problems Using Variation of Parameters Method

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Abstract

The Variation of Parameters Method (VPM) use to solve initial and boundary value problems of vary objective nature. The logical results are calculated in terms of convergent series with easily computable components. The Variation of Parameters Method (VPM) is use without any, transformation or restrictive assumptions, perturbation and discretization is free from round off errors and calculation of the so called Adomian's polynomials. The recommended algorithm is tested on higher dimensional initial and boundary value problems, Helmholtz equations and Boussinesq and nonlinear boundary value problems of various orders. The Numerical results tell the complete consistency of the proposed in ternVPM.

1. Introduction

A number of methods including exp –function, sink-Galerkin, perturbation, homotopy perturbation, variational iteration, finite difference, polynomial spline and Adomian's decomposition are use to solve initial and boundary value problems [1-49] and references there in.. These initial and boundary value problems vary physical nature problems and an integral part in the study of engineering, shallow water wave,applied sciences, theory of stellar structure, physics, fluid mechanics and astrophysics. Motivated and inspired by the ongoing research in this way, we apply Variation of Parameters Method (VPM) [11, 13, 32, 36, 37] for solving a large class of initial and boundary value problems. The designed VPM is tested on higher dimensional initial and boundary value problems, Helmholtz equations, Boussinesq and nonlinear boundary value problems of various orders. The suggested method is free from round off errors, calculation of the so-called Adomian's polynomials, perturbation, linearization, discretization and uses only the initial conditions which are easier to implement and reduces the computational work while still maintaining a higher level of accuracy. It is worth mentioning that Ma *et al.* [11-13] used Variation of Parameters Method (VPM) for solving involved non-homogeneous partial differential equations and obtained solution formulas helpful in constructing the existing solutions coupled with a number of other new solutions including rational solutions, solitons, positions, negatons, breathers, complextions and interaction solutions of the KdV equations. It has been observed that Ma's VPM [11-13] is much better as compare to the above mentioned algorithms. Firstly, it does not require the small parameter assumption which is a major drawback in the traditional perturbation methods.

Moreover, Ma's VPM is more reliable than Homotopy Analysis Method (HAM) which is a generalized Taylor series method, gives an infinite series solution and is coupled with all the deficiencies and limitations of this technique to have practical examples. Moreover, such schemes (HAM) are not compatible to cope with the secular terms arising in the higher-order approximate solutions, whereas Variation of Parameters Method (VPM) gives an asymptotic solution with few terms. The Ma's VPM does not require the calculation of so-called Adomian's polynomials and hence is a better option as compare to decomposition method. Moreover, Va riation of Parameters Method (VPM) is brief, concise and more generalized than the above mentioned technique and does not even require any unrealistic assumptions which ruin the basic physical structure of the nonlinear problems. Numerical results coupled with the graphical representations explicitly reveal the complete reliability of these algorithms. It needs to be highlighted that Ma *et al.* [11-26] also presented number of other revolutionary techniques for solving diversified nonlinear problems of physical nature.

2. Mathematical Formulation of Helmholtz Equation

The partial differential equation

$$\sum_{k=1}^{n} \frac{\partial^2 \mu}{\partial x^2} + cu = 0$$

Where c is a constant. The Helmholtz equation is used in the study of stationary oscillating

Processes. If $c = 0$ the Helmholtz equation becomes the Laplace equation. If a function f appears on the right-hand side of the Helmholtz equation, this equation is known as the inhomogeneous Helmholtz equation.

The usual boundary value problems (Dirichlet, Neumann and others) are posed for the Helmholtz equation, which is of elliptic type, in a bounded domain. A value of c for which a solution of the homogeneous Helmholtz equation not identically equal to zero and satisfying the corresponding homogeneous boundary condition exists, is called an eigen value of the Laplace operator (of the corresponding boundary value problem). In particular, for the Dirichlet problem all eigen values are positive, and for the Neumann problem they are all non-negative. It is known that a solution of the boundary value problem for the Helmholtz equation is not unique for a value of c which coincides with an eigen value. If, on the other hand, the value of c is not an eigen value, the uniqueness theorem is valid. Boundary value problems for the Helmholtz equation are solved by the ordinary methods of the theory of elliptic equations (reduction to an integral equation, variational methods, methods of finite differences).

In the case of an unbounded domain with compact boundary one can state exterior boundary value problems for the Helmholtz equation; if $c < 0$, these have a unique solution which tends to zero at infinity. If $c > 0$, solutions tending to zero at infinity are usually not unique. In such cases additional restrictions are imposed to obtain a unique solution.

The following mean-value formula is valid for a solution of the Helmholtz equation which is regular in a domain G:

$$\frac{1}{mes\Omega} \int u d\sigma = u(x_\circ)\Gamma\left(\frac{n}{2}\right) 2^{\frac{n-2}{2}} (r\sqrt{c})^{1-\frac{n}{2}} J_{\frac{n-2}{2}}(r\sqrt{c})$$

where Ω is the sphere of radius r with centre at a point $x_\circ$, which must lie entirely within G, and $J_v(x)$ is the Bessel function of order v.

The equation was studied by H. Helmholtz in 1860, who obtained the first theorems on the solution of boundary value problems for this equation.

3. Solution Procedure

Consider the following Helmholtz equation

$$\frac{\partial^2 \mu(\phi, y)}{\partial \phi^2} + \frac{\partial^2 \mu(\phi, y)}{\partial y^2} - \mu(\phi, \psi) = 0$$

Initial Condition

$$\mu(0, y) = y, \mu\phi\,(0, y) = y + \cosh(y)$$

The exact solution of the problem is

$$\mu(\phi, y) = ye^{\phi} + \phi\cosh(y)$$

$$(x, y) = A_{1(\phi,y)} + A_{2(\phi,y)} + \int_0^{\phi} (\phi - s)(\,\mu_n(\phi, y) - \frac{\partial^2 \mu(\phi, y)}{\partial y^2})\, ds$$

Using the initial conditions, we have

$$A_1(\phi, y) = y, A_2(\phi, y) = y + \cosh(y)$$

where

$$\mu_0(\phi, y) = y + \phi y + \phi cosh y$$

$$put\ n = 0$$

$$\mu_1 = y + \phi y + \phi coshy(y) + \int_0^{\phi} (\phi - s)(y + \phi y + \phi cosh(y) - \frac{\partial^2}{\partial y^2}(y + \phi y + \phi cosh(y)\big)ds$$

$$= y + \phi y + \phi coshy(y) + \int_0^{\phi} (\phi - s)(y + \phi y + \phi cosh(y) - \phi coshy(y))ds$$

$$= y + \phi y + \phi coshy(y) + \int_0^{\phi} (\phi - s)(y + sy)ds$$

$$= y + xy + xcoshy(y) + \int_0^{x} (xy + sxy - sy - s^2 y)ds$$

$$\mu_1 = y + xy + xcoshy(y) + x^2 y + \frac{x3}{2} y - \frac{x^2}{2} y - \frac{x^3}{3} y$$

$$\mu_1 = y + xy + xcoshy(y) + \frac{x2}{2} + \frac{x3}{6} y$$

$$\mu_2 = y + xy + xcoshy(y) + \int_0^{x} (x - s)(y + xy + xcosh(y) + \frac{x2}{2} + \frac{x3}{6} y - \frac{\partial^2}{\partial y^2}(y + xy + xcosh(y)) + \frac{x^2}{2} y + \frac{x3}{6} y))\ ds$$

$$= y + xy + xcoshy(y) + \int_0^{x} (x - s)(y + xy + xcosh(y) + \frac{x2}{2} + \frac{x3}{6} y - xcoshy(y))ds$$

$$= y + xy + xcoshy(y) + \int_0^{x} (x - s)(y + sy + + \frac{s2}{2} + \frac{s3}{6} y)ds$$

$$= y + xy + xcoshy(y) + \int_0^{x} \left(xy + sxy + \frac{s^2}{2} xy + \frac{s3}{6} xy - sy - s^2 y - \frac{s3}{2} y - \frac{s4}{6} y \right) ds$$

$$= y + xy + xcoshy(y) + x^2 y + \frac{x3}{2} y + \frac{x4}{6} y - \frac{x2}{2} y - \frac{x3}{3} y - \frac{x4}{8}$$

$$= y + xy + xcoshy(y) + \frac{x2}{2} y + \frac{x3}{6} y + \frac{x4}{24} y$$

$$\mu_3 = y + xy + xcoshy(y) + \frac{x2}{2} y + \frac{x3}{6} y + \frac{x4}{24} y + \frac{x5}{120} y$$

Therefore

$$\mu(x, y) = y + xy + xcoshy(y) + \frac{x2}{2} y + \frac{x3}{6} y + \frac{x4}{24} y + \frac{x5}{120} y \ldots \ldots$$

$$= xcoshy(y) + y(1 + \frac{x2}{2!} y + \frac{x3}{3!} y + \frac{x4}{4!} y + \frac{x5}{5!} y \ldots .)$$

$$= xcoshy(y) + ye^x$$

$$\mu(x, y) = xcoshy(y) + ye^x$$

Which is the exact solution.

Table 1. *Comparision between Exact and Approximate Solutions.*

i	X(i)	U(X(i))	V(X(i))	abs(U(X(i))-V(X(i)))
0	0.00	0.0000000000	0.0000000000	0.0000000000e+00
1	0.10	0.0198100000	0.0198100000	0.0000000000e+00
2	0.20	0.0771200000	0.0771200000	0.0000000000e+00
3	0.30	0.1662300000	0.1662300000	0.0000000000e+00
4	0.40	0.2790400000	0.2790400000	0.0000000000e+00
5	0.50	0.4062500000	0.4062500000	0.0000000000e+00
6	0.60	0.5385600000	0.5385600000	0.0000000000e+00
7	0.70	0.6678700000	0.6678700000	0.0000000000e+00
8	0.80	0.7884800000	0.7884800000	0.0000000000e+00
9	0.90	0.8982900000	0.8982900000	0.0000000000e+00
10	1.00	1.0000000000	1.0000000000	0.0000000000e+00

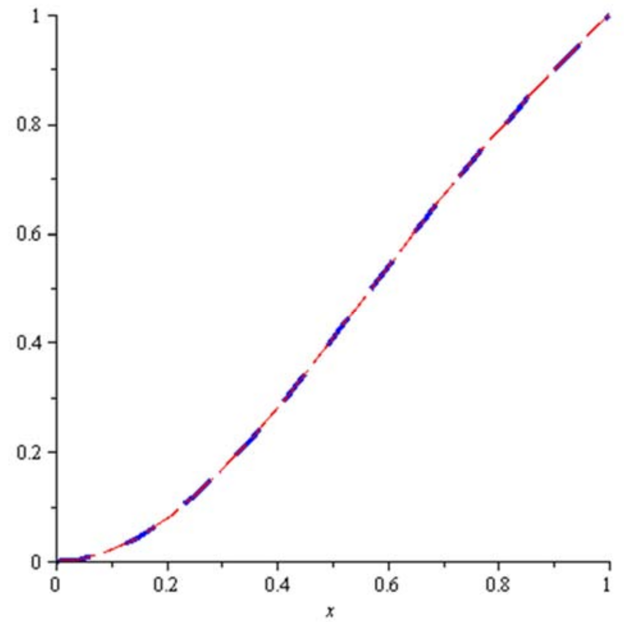

Figure 1. *Graphical Representations: Comparision between Exact and Approximate Solutions.*

4. Conclusion

The Variation of Parameters Method (VPM) use to solve initial and boundary value problems of vary objective nature. The planned method is use without using perturbation, discretization or restrictive assumptions, linearization and is free from round off errors and calculation of the so called Adomian's polynomials. It is concluded that the VPM is very influential and capable in finding the logical solutions for a large class of boundary value problems and it is another method for solving nonlinear initial and boundary value problems.

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