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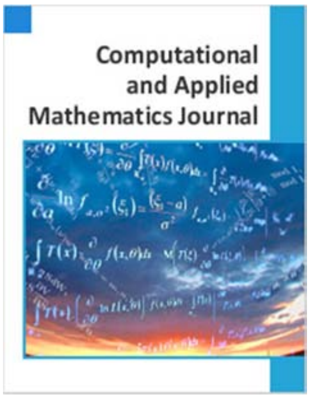

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Second Order Iterative Techniques for Boundary Value Problems and Fredholm Integral Equations

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Abstract
Comparison between the performance of the second and first degree stationary iterative techniques is performed on two representations to two point boundary value problems of the second and fourth orders. The numerical treatment of the Fredholm integral equation representation for the second and fourth order boundary value problems (BVP) has illustrated the effective use of their integral representation. The finite difference method with the same accuracy is employed to construct linear systems from the differential or the equivalent Fredholm form. Second degree linear stationary iterative methods are extensively used. The second degree Gauss-siedle (S. G. S) method is presented, by measuring the asymptotic rates of convergence of the sequence; we are able to determine when the Gauss-Seidle second-degree iterative method is superior to its corresponding first-degree one. Two numerical examples are considered, one of them is of the second degree BVP and the other is fourth order BVP. All calculations are done with the help of computer algebra system (MATHEMATICA 10.2).

1. Introduction

Many real-life, engineering phenomena, population dynamics, diffusion, neurophysiology and feedback control theory can be formulated mathematically as differential or integral equations. Sometimes, the mathematical models appear in biology, chemistry, physics and engineering have to consider the memory behaviors, [1, 2, 3].

Differential equations models consider only the local interactions while integral equations models involve local and global interactions. Some physical situations can be described by differential and integral equations at the same time and this introduce the question about the relation between differential and integral equations. Mathematically it is well known that from any differential equation one can obtains a corresponding integral equation, but the inverse is not true in general [1]. Closed form analytical solutions are available for limited classes of differential and integral equations. The alternative approach to obtain or even approximate solutions is the use of numerical techniques. In recent years, numerous works have been focusing on the development of more advanced and efficient methods [4].

Solution of linear algebraic systems [5] has its great importance even in the solution of nonlinear problems; it is the final stage in the numerical treatment in many techniques.

We discussed the linear models which can be formulated in both integral and differential equations which reduce large linear algebraic systems, and the problem is reduced to that of the efficient use of iterative techniques for solving these systems. Because, differential equations give rise to large sparse linear systems, these sparse systems in many cases tend to be badly conditioned. By contrast, integral equations give rise to dense large well-conditioned coefficient matrices [6]. Integral equation has advantages; it includes the boundary conditions which the problem must satisfy in addition to the convenient theory of existence and uniqueness available [3].

The algebraic system corresponding to two well-known two point BVP and its integral representations are considered: the first one is the second degree BVP.

$$y''(x) + \lambda y(x) = f(x);\ (a < x < b), y(a) = \alpha, y(b) = \beta. \quad (1)$$

And its equivalent second kind integral equation

$$y(x) = \alpha + \frac{(\beta-\alpha)}{(b-a)}(x-a) + \frac{\lambda}{(b-a)}\int_a^b k(x,t)y(t)dt - \frac{1}{(b-a)}\int_a^b k(x,t)f(t)dt \quad (2)$$

Where

$$k(x,t) = \begin{cases}(t-a)(b-x)\,;\,(t \le x),\\(x-a)(b-t)\,;\,(x \le t).\end{cases}$$

The second is the fourth order two-point BVP [3]:

$$y^{(4)}(x) = \lambda y(x), (0 \le x \le 1),\ y(0) = y''(0) = \sigma, y(1) = y''(1) = \tau \quad (3)$$

And its equivalent second kind Fredholm integral equation

$$y(x) = \sigma\left(1 + \frac{x^2}{2}\right) + \left(\frac{5\tau - 8\sigma}{6}\right)x + \frac{(\tau-\sigma)}{6}x^3 + \lambda\int_0^1\int_0^1\{k(x,s)k(s,t)ds\}y(t)dt \quad (4)$$

Where

$$k(x,t) = \begin{cases}t(1-x)\,;\,(t \le x),\\x(1-t)\,;\,(x \le t).\end{cases}$$

Our treatment depends mainly on transforming the given Boundary value problem into an equivalent Fredholm integral equation [7]. The finite difference method is employed in the replacement of the given differential or integral system [4, 8, 9] by a corresponding algebraic system, the trapezoidal numerical integration technique is used in approximating the integrals. Comparison of the performance of the first and the second degree Gauss-Siedle iterative methods, [10, 11, 12, 13 & 14] are introduced.

1.1. The Central Finite Difference Approximations

The basic idea of the finite difference approximation is the replacement of derivatives or integrals by difference approximations. The region [a, b] of the differential or integral equations is super imposed with a uniform mesh with mesh size $h > 0$, and the grid points are defined by [4, 8, 15, 16]

$$x_j = a + jh; j = 0,1,\cdots,n; h = \frac{b-a}{n}$$

Let y_j denote the approximate solution of the equation at any point x_j , i.e. $y(x_j) = y_j$ then the second order derivative can be approximated by the central difference

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \quad (5)$$

Also, the central difference for fourth order derivative

$$y_i^{(4)} = \frac{y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}}{h^4} \quad (6)$$

1.2. The Trapezoidal Rule

It is well known that the value of a definite integral can be approximated by a combination of functional values of the integrand with different methods depending on the required accuracy and the grid points used. The trapezoidal rule is one method which uses only the end points of the interval of integration and gives second order accuracy and it takes the form [9]:

$$\int_a^b f(x)dx = \frac{h}{2}\{f(a) + f(b)\} - \frac{(b-a)^3}{12}f''(\xi)$$

The composite form of the trapezoidal rule takes the form

$$\int_a^b f(x)dx = \frac{h}{2}\left\{f(x_0) + 2\sum_{j=1}^{n-1} f(x_j) + f(x_n)\right\} - \frac{(b-a)\,h^2}{12}f''(\xi) \quad (7)$$

Where $-\frac{(b-a)h^2}{12}f''(\xi)$ is the error term, and $\xi \in (a,b)$.

2. Basic Iterative Methods

There are numerous analytical and numerical methods for the solution of a linear system, including Gauss elimination method, Crout’s method and Cholesky’s method, which employ LU-decomposition method.

The general form of a linear algebraic system

$$Ax = y_0 \quad (8)$$

Where(A), is a real nonsingular coefficients matrix and y_0 is a given column vector. First order stationary iterative methods take the form

$$x^{(n+1)} = Bx^{(n)} + k \quad (9)$$

Where, B is the iteration matrix, $k = A_1^{-1}y_0$ and A_1 is called the splitting matrix. Usually, we write $A = D - L - U$, where D, the diagonal part of A, $-L$ and $-U$ are the strictly lower and the strictly upper triangular parts of A, [5, 14, 17, 18].

2.1. Jacobi Method

The simplest iterative method for solving the linear system (8) is Jacobi method, Jacobi method is classified as first degree method and takes the form (9) if $A_1 = D$ and $B = D^{-1}(L + U)$as

$$x^{[n+1]} = Bx^{[n]} + D^{-1}y_0 = D^{-1}(L + U)x^{[n]} + D^{-1}y_0 \quad (10)$$

2.2. Gauss-Seidle Method

From the computational point of view Gauss-Seidle method is known as a modification of Jacobi method. Historically, Gauss introduced his method when he was working in least squares problem, in 1823, while Jacobi work appeared in 1853. Gauss-Seidle idea depends on the use of the most recent calculated values. The first degree Gauss-Seidle method for system (8) given from Eq. (9) with $A_1 = D - L$ and its iteration matrix $B = (D - L)^{-1}U$. It can be written in the form

$$x^{[n+1]} = Bx^{[n]} + (D - L)^{-1}y_0 = (D - L)^{-1}Ux^{[n]} + (D - L)^{-1}y_0 \quad (11)$$

2.3. Successive Overrelaxation (SOR) Method

The Successive Overrelaxation approach, the SOR method, generalizes the Gauss-Seidle by introducing a relaxation parameter ω. The first degree SOR iterative method can be obtained from (9) with $A_1 = (D - \omega L)^{-1}$and its iteration matrix $B = (D - \omega L)^{-1}\big((1 - \omega)D + \omega U\big); \omega \in (0,2)$, or can be written in the explicit form as

$$x^{[n+1]} = Bx^{[n]} + (D - \omega L)^{-1}y_0 = (D - \omega L)^{-1}\big((1 - \omega)D + \omega U\big)x^{[n]} + (D - \omega L)^{-1}y_0 \quad (12)$$

Definition: the spectral radius of a matrix A, denoted by $\rho(A)$ is given by$\rho(A) = \text{Max}\ \{|\lambda_i|;\ \lambda_i$ is an eigenvalue of $A\}$

2.4. The Asymptotic Rate of Convergence R_∞

The general theorem of iterative method states that Eq. (9) converges if and only if

$$\rho(B) < 1,$$

The standard measurement for how fast an iterative method convergence is referred to as the asymptotic rate of convergence. The asymptotic rate of convergence is adopted as a standard measure of the speed of the iterative method. The asymptotic rate of convergence of an iterative method $R_\infty(B)$ can be calculated from the formula

$$R_\infty(\mathrm{B}) = -\log_{10}\rho(B) \quad (13)$$

3. Second Degree Iterative Methods

Our aim in this work is to compare between the first degree iterative methods (two-part splitting given in the previous section and the second degree stationary iterative methods (three-part splitting) in the solution of the linear system arising from the BVP and its equivalent Fredholm integral equations.

Definition: let $A = A_1 + A_2'$ be a two-part splitting for A, $A = A_1 + A_2 + A_3$ be a three-part splitting for A and let vector y_0 be fixed. Then for any $x_0 = x_0'$, the two-part sequence $\{x_0', x_1', x_2', \cdots, x_k', \cdots\}$ is defined iteratively by

$$A_1x_{k+1}' + A_2'x_k' = y_0 \quad (14)$$

The three-part sequence $\{x_0, x_1, x_2, \cdots, x_k, \cdots,\}$ is defined iteratively by

$$A_1x_{k+2} + A_2x_{k+1} + A_3x_k = y_0 \quad (15)$$

3.1. Relation Between Two-Part and Three-Part Splitting Sequences

Now we want to see the relation between the two-part and three-part splitting. The three-part splitting sequence (15) can be written in the form

$$\begin{bmatrix} x_{k+2} \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} -A_1^{-1}A_2 & -A_1^{-1}A_3 \\ I & 0 \end{bmatrix} \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} + \begin{bmatrix} A_1^{-1}y_0 \\ 0 \end{bmatrix} \text{ or } Z_{k+1} = B_2Z_k + Y_0 \quad (16)$$

Where $x_k \to x$ if and only if $Z_k = \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} \to Z = \begin{bmatrix} x \\ x \end{bmatrix}$, moreover

$$Z_k - Z = {B_2}^k(Z_0 - Z)$$

for two-part splitting (14), we have similarly

$$x_k - x = B^k(x_0 - x)$$

Where B_2 is the iteration matrix for three-part splitting given as

$$B_2 = \begin{bmatrix} -A_1^{-1}A_2 & -A_1^{-1}A_3 \\ I & 0 \end{bmatrix} \quad (17)$$

Where I is the identity matrix, B is the iteration matrix for two-part splitting given as

$$B = -A_1^{-1}A_2' \quad (18)$$

The asymptotic rate of convergence for two-part (R') and three-part splitting (R) given as:

$$R' = -\log_{10}(B)\ ;\ R = -\log_{10}(B_2)$$

3.2. Construction of A_3 and the Relation Between $\rho(B)$ and $\rho(B_2)$

The coefficient matrix can be written as $A = A_1 + (A_2 + A_3) = A_1(I - B)$ [10, 11], then for any complex $s \neq -1$, where s will defined in the next theorem depending on $\sigma(B)$ (the eigenvalues of the iteration matrix B we can define A_3 as a function of s as follow

$$A_3(s) = \frac{-s}{s+1}(sA_1 + B) = \frac{s}{s+1}A - sA_1 \qquad (19)$$

Then with the three-part iteration matrix $B_2 = B_2(s)$ the three-part sequences (15) can be written in the form

$$x_{k+2} = -A_1^{-1}A_2x_{k+1} - A_1^{-1}A_3x_k + A_1^{-1}y_0$$

But $-A_1^{-1}A_2 = (1-s) - \frac{A_1^{-1}A}{s+1}$ and $-A_1^{-1}A_3 = s - \frac{s}{s+1}A_1^{-1}A$ then the three-part sequence for linear system (8) as a function of s takes the form

$$x_{k+2}(s) = (B - I)\left(\frac{1}{s+1}x_{k+1}(s) + \frac{s}{s+1}x_k(s)\right) + (1-s)x_{k+1}(s) + sx_k(s) + A_1^{-1}y_0 \qquad (20)$$

Where, $B - I = -A_1^{-1}A$. The two-part sequence for the linear system (8) is

$$x'_{k+1} = Bx'_k + A_1^{-1}y_0 \qquad (21)$$

Although the rate of convergence in the three-part sequence is better than the rate of convergence in two-part sequence, we observed that in case of three-part sequence (20) the equation doesn't have new parameters comparing with equation (21) in case of two-part sequence, both equations has the parameters B and A_1^{-1}.

This work is concerned with the Gauss-Seidle iterative method. The iteration matrix B and the splitting matrix A_1 is described in subsection (2.2) Eq. (11).

Why we used Gauss-Seidle technique? Because for arbitrary $x^{(0)}$ (initial iteration), we known that Gauss-Seidle is the improvement of Jacobi and according to SOR technique, the numerical results proved that the only best value of ω was unity, i.e., while the Gauss-Seidle iterative technique lends itself to acceleration by second degree iterative methods, but SOR method does not [12, 19].

The following theorem calculates the optimal value of s which makes the three-part splitting (second degree iterative methods) converges faster than the two-part splitting or (first iterative methods) or which makes $R > R'$.

Theorem 3.1

Given $A = A_1(I - B)$, where for real $\alpha, \beta \in \sigma(-B)$, we have $\sigma(-B) \subset [\alpha, \beta]$. thus

$$\rho(-B) = \max\{|\alpha|, |\beta|\}.$$

We suppose $-1 < \alpha \leq \beta \leq 3$, let the midpoint be denoted

$$m = \frac{1}{2}(\alpha + \beta)$$

Then the following table is valid for the optimal $s = s_0$ generating

$$\min \rho(B_2(s)) = \rho(B_2(s_0))$$

Table 1. *Used to compute the optimal value for (S) depending on the spectrum of (B).*

$-1 < \alpha \leq \beta \leq 3,\ m = \frac{1}{2}(\alpha+\beta) \neq 0,$		
$m > 0 \rightarrow \rho(B) = \beta$	If $m^2 \leq -\alpha$ Then $s_0 = m$ $\rho(B_2(s_0)) = \frac{m-\alpha}{m+1}$	If $m^2 \geq -\alpha$ Then $s_0 = -1 + \sqrt{1+\beta}$ $\rho(B_2(s_0)) = -1 + \sqrt{1+\beta}$
$m > 0 \rightarrow \rho(B) = -\alpha$	$m \geq -1 + \sqrt{1+\alpha}$ Then $s_0 = m$ $\rho(B_2(s_0)) = \frac{m-\alpha}{m+1}$	If $m \leq -1 + \sqrt{1+\alpha}$ Then $s_0 = -1 + \sqrt{1+\alpha}$ $\rho(B_2(s_0)) = 1 - \sqrt{1+\alpha}$

In all four cases in table

$$\rho(B_2(s_0)) < \rho(B)$$

4. Initiations

In the two-part splitting (21) each vector depends on the previous vector only so we choose one initial vector call it $x^{(0)}$. But in the three-part splitting (20) each vector depends on the two previous vectors so we need two initial vectors $x^{(0)}$& $x^{(1)}$. $x^{(0)}$ in most cases is usually taken as zero but $x^{(1)}$ can be chosen in the following forms:

- i Choosing $x^{(1)} = x^{(0)} = 0$ according to [11] in this case we make an improvement.
- ii Choosing $x^{(0)} = 0$ and $x^{(1)} = A_1^{-1}y_0$ which is the first iteration of Jacobi, in this case an improvement more than one iteration was gotten comparing with the choice (i).
- iii Choosing $x^{(0)} = 0$ and according to [14]

$$x^{(1)} = \frac{1}{2-(\beta+\alpha)}[2B - (\beta+\alpha)I]x^{(0)} + \frac{2}{2-(\beta+\alpha)}A_1^{-1}y_0 \Rightarrow x^{(1)} = \frac{2}{2-(\beta+\alpha)}A_1^{-1}y_0 \qquad (22)$$

In this case we get an improvement better than the choice (i) and (ii) as we will see in the numerical examples later.

5. Numerical Examples

Example (1)

Consider the second order B. V. P, [20]

$$-y''(x)+\pi^2 y(x)=2\,\pi^2\,Sin(\pi\,x)\,,0\le x\le 1;\ y(0)=y(1)=0 \qquad (23)$$

Whose exact solution is

$$y(x)=\sin(\pi\,x) \qquad (24)$$

The Fredholm integral equation is

$$y(x)=2\,sin\,\pi x-\pi^2\int_0^x t(1-x)\,y(t)dt-\pi^2\int_x^1 x(1-t)\,y(t)dt \qquad (25)$$

It is an easy task to see that this integral equation (25) satisfies the boundary conditions in (23). Moreover, the closed form solution (24) satisfies both the differential and the integral equation.

Using finite difference scheme with $h=0.1$ and with $x_i=i\,h, i=1(1)9,\ y(x_i)=y_i$, one can derive a linear system of algebraic equations, corresponding to the differential equation, with coefficient matrix

$$\begin{pmatrix} 2.098696 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 2.098696 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 2.098696 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 2.098696 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 2.098696 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 2.098696 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 2.098696 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2.098696 & -1\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2.098696 \end{pmatrix} \qquad (26)$$

Using the trapezoidal rule with $h=0.1$ we obtain a functional relation which is satisfied at each point of the grid points x_i, $x_i=i\,h, i=1(1)9,\ y(x_i)=y_i$ one can derive a linear system with coefficient matrix in the form (27).

Table 1. *The performance of S.G.S method using the rate of convergence for Ex. (1).*

h	**Diff. Form** R' $\rho(B)=0.82$	R $[s_0=-0.41,\ \rho(B_2)=0.697]$	**Con. Ratio** R/R'	**Integ. Form** R' $\rho(B)=0.137$	R $[s_0=0.0877,\ \rho(B_2)=0.045]$	**Con. Ratio** R/R'
0.1	0.085426	0.156776	1.82	0.8639	1.34588	1.55

Table 2. *Comparison between convergence rate of integral and differential systems using F.G.S and S.G.S methods for Ex. (1).*

Step Size	**F.G.S** R'$_{diff.}$	R'$_{integ.}$	Con. Ratio= R'$_{integ.}$/R'$_{diff.}$	**S.G.S** R$_{diff.}$	R$_{integ.}$	Con. Ratio=R$_{integ.}$/R$_{diff.}$
0.1	0.0854264	0.8639	10.11	0.1568	1.34588	8.58

Solve the two systems (26) & (27) using first degree Gauss-Seidle (F.G.S) and second degree Gauss-Siedle (S.G.S) iterative methods, the results are shown in table 2 and table 3.

$$\begin{pmatrix} 1.088826 & 0.0789568 & 0.0690872 & 0.0592176 & 0.049348 & 0.0394784 & 0.0296088 & 0.0197392 & 0.0098696\\ 0.078956 & 1.157914 & 0.138174 & 0.118435 & 0.098696 & 0.0789568 & 0.0592176 & 0.0394784 & 0.0197392\\ 0.069087 & 0.138174 & 1.207262 & 0.177653 & 0.148044 & 0.118435 & 0.0888264 & 0.059217 & 0.0296088\\ 0.059217 & 0.118435 & 0.177653 & 1.23687 & 0.197392 & 0.157914 & 0.118435 & 0.0789568 & 0.0394784\\ 0.049348 & 0.098696 & 0.148044 & 0.197392 & 1.24674 & 0.197392 & 0.148044 & 0.098696 & 0.049348\\ 0.039478 & 0.0789568 & 0.118435 & 0.157914 & 0.197392 & 1.23687 & 0.177653 & 0.118435 & 0.0592176\\ 0.029608 & 0.0592176 & 0.0888264 & 0.118435 & 0.148044 & 0.177653 & 1.207262 & 0.138174 & 0.0690872\\ 0.019739 & 0.0394784 & 0.059217 & 0.0789568 & 0.098696 & 0.118435 & 0.138174 & 1.157914 & 0.0789568\\ 0.009869 & 0.0197392 & 0.0296088 & 0.0394784 & 0.049348 & 0.0592176 & 0.0690872 & 0.0789568 & 1.0888264 \end{pmatrix} \qquad (27)$$

Example (2)

The normal modes of free flexural vibration of a thin, uniform rod of unit length are governed approximately by the differential equation, [7]

$$y^{(4)}(x)-\lambda y(x)=0; y(0)=y''(0)=1, y(1)=y''(1)=e, 0\le x\le 1, \qquad (28)$$

whose exact solution when $\lambda=1$ is

$$y(x)=e^x;\ \ 0\le x\le 1, \qquad (29)$$

Where, $y(x)$ represents the transverse displacement of the centroid of the cross-section of the rod, at position x, from its equilibrium position, and λ is proportional to σ^2 where σ, the frequency of vibration, is not known in advance.

Equation (28) is equivalent to system of second order differential equations, as follow, let $y''\,(x)=-\psi(x)$, then

$$y''(x) = -\psi(x); \psi''(x) = -y(x), 0 < x < 1, \psi(0) = -1;\ \psi(1) = -e, y(0) = 1, y(1) = e. \quad (30)$$

Also the Fredholm integral form to (28) is

$$y(x) = f(x) + \int_0^x t\left(\frac{x^2}{2} - \frac{5x^3}{6} + \frac{x^4}{3}\right) y(t)dt + \int_x^1 (1-t)\left(\frac{x}{6} - \frac{x^3}{2} + \frac{x^4}{3}\right) y(t)dt, \quad (31)$$

$$f(x) = 1 + \frac{(5e-8)}{6}x + \frac{x^2}{2} + \frac{(e-1)}{6}x^3 \quad (32)$$

***Table 3.** Illustrate the best choice for the initial $x^{(1)}$ in Ex(1).*

x	y_{ext}	Solution of Differ. form			
		F.G.S.	S.G.S. $x^{(0)} = 0$ $x^{(1)} = 0$	S.G.S. $x^{(0)} = 0$ $x^{(1)} = A_1^{-1} y_0$	S.G.S. $x^{(0)} = 0$ $x^{(1)} = A_1^{-1} y_0 * \frac{2}{2-(\beta+\alpha)}$
		(73) iterations	(63)iterations	(62)iterations	(58)iterations
0.1	0.309017	0.31029	0.31029	0.31029	0.31029
0.2	0.587785	0.5902	0.5902	0.5902	0.5902
0.3	0.809017	0.81235	0.81235	0.81235	0.81235
0.4	0.951057	0.95497	0.95497	0.95497	0.95497
0.5	1	1.0041	1.0041	1.0041	1.0041
0.6	0.951057	0.95497	0.95497	0.95497	0.95497
0.7	0.809017	0.81235	0.81235	0.81235	0.81235
0.8	0.587785	0.5902	0.5902	0.5902	0.5902
0.9	0.309017	0.31029	0.31029	0.31029	0.31029

It is easy to see that integral equation (31) satisfies the boundary conditions in (28). Moreover the closed form solution (29) satisfies both the differential equation and the Fredholm integral equation.

Using finite difference method one can derive a linear system of algebraic equations, corresponding to the differential equation. By using different values for the step size, we have the following:

i let $h = 0.2$, the finite difference scheme for (28)

$$y_{i-2} - 4y_{i-1} + 5.9984 y_i - 4y_{i+1} + y_{i+2} = 0. \quad (33)$$

With, $i = 1(1)4, y(0) = y''(0) = 1$, and $y(1) = y''(1) = e$, we obtain a linear system with coefficient matrix

$$A = \begin{pmatrix} 2.9984 & -3 & 1 & 0 \\ -4 & 5.9984 & -4 & 1 \\ 1 & -4 & 5.9984 & -4 \\ 0 & 1 & -3 & 2.9984 \end{pmatrix} \quad (34)$$

Taking $h = 0.2$ and the trapezoidal rule in (31) we obtain a functional relation which is satisfied at each point of the grid points x_i, with $x_i = i\,h, i = 1(1)4, y(x_i) = y_i$, and the linear system with coefficient matrix

$$A = \begin{pmatrix} 0.99733 & -0.003584 & -0.0023893 & -0.001195 \\ -0.001408 & 0.996 & -0.003456 & -0.001728 \\ -0.001728 & -0.003456 & 0.996 & -0.001408 \\ -0.001195 & -0.0023893 & -0.003584 & 0.99733 \end{pmatrix} (35)$$

The results of solution of the two systems whose coefficient matrices (34) & (35) using F.G.S and S.G.S iterative methods given in table 5.

ii when $h = 0.1$, the finite difference scheme for (6) gives

$$y_{i-2} - 4y_{i-1} + 5.9999 y_i - 4y_{i+1} + y_{i+2} = 0. \quad (36)$$

Taking $i = 1(1)9$ one can derive a linear system of algebraic equations with coefficients matrix in the form (37).

Using $h = 0.1$ and the trapezoidal rule in (31) lead to a linear system of algebraic equations with coefficients matrix, in the form (38).

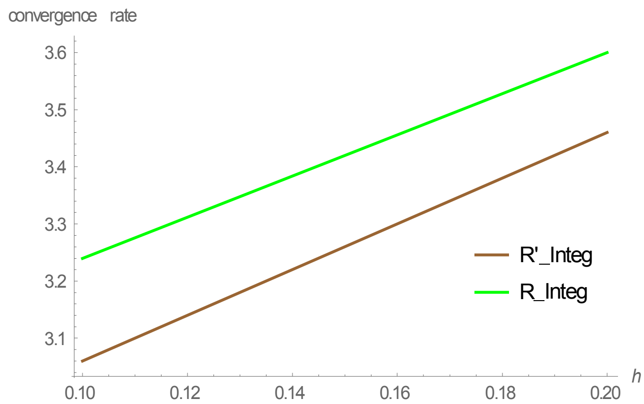

***Figure 1.** Shows the priority of S.G.S than F.G.S in integral form of Ex. (2).*

$$\begin{pmatrix} 2.9999 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 5.9999 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 5.9999 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 5.9999 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 5.9999 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -4 & 5.9999 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 5.9999 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 & 5.9999 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 2.9999 \end{pmatrix} \quad (37)$$

$$\begin{pmatrix} -0.99925 & 0.001296 & 0.001134 & 0.000972 & 0.00081 & 0.000648 & 0.000486 & 0.000324 & 0.000162 \\ 0.000139 & -0.99867 & 0.002091 & 0.001792 & 0.001493 & 0.001195 & 0.000896 & 0.000597 & 0.000299 \\ 0.000252 & 0.000504 & -0.99825 & 0.002352 & 0.00196 & 0.001568 & 0.001176 & 0.000784 & 0.000392 \\ 0.000352 & 0.000704 & 0.001056 & -0.998 & 0.00216 & 0.001728 & 0.001296 & 0.000864 & 0.000432 \\ 0.000417 & 0.000833 & 0.00125 & 0.001667 & -0.99792 & 0.001667 & 0.00125 & 0.000833 & 0.000417 \\ 0.000432 & 0.000864 & 0.001296 & 0.001728 & 0.00216 & -0.998 & 0.001056 & 0.000704 & 0.000352 \\ 0.000392 & 0.000784 & 0.001176 & 0.001568 & 0.00196 & 0.002352 & -0.99825 & 0.000504 & 0.000252 \\ 0.000298 & 0.000597 & 0.000896 & 0.001195 & 0.001493 & 0.001792 & 0.002091 & -0.99867 & 0.000139 \\ 0.000162 & 0.000324 & 0.000486 & 0.000648 & 0.00081 & 0.000972 & 0.001134 & 0.001296 & -0.99925 \end{pmatrix} \quad (38)$$

The results of solving the two systems whose coefficient matrices are (37) & (38) using F.G.S and S.G.S methods are given in table 7.

Table 4. *The number of iterations using F.G.S and S.G.S in both differential and integral systems for Ex.(2), h=0.2.*

x	y_{ext}	**Differential form**		**Integral form**		
		F.G.S	**S.G.S**	**F.G.S (3) iterations**	**S.G.S** $s_0 = -0.0000937$	
					$x^{(1)} = 0$	$x^{(1)} = \frac{2}{2-(\beta+\alpha)} A_1^{-1} y_0$
					(3)iterations	**(2) iterations**
0.2	1.2214	Divergent	Divergent	1.22428	1.22428	
0.4	1.49182			1.48891	1.48891	
0.6	1.82212			1.81867	1.81867	
0.8	2.22554			2.22963	2.22963	

Solving the system of second order differential equations (30)

(i) when $h = 0.2$ and using finite difference method in equation (30) we obtain

$$-y_{i-1} + 2y_i - y_{i+1} - 0.04\psi_i = 0, y(0) = 1, y(1) = e - \psi_{i-1} + 2\psi_1 - \psi_{i+1} - 0.04y_i = 0, \psi(0) = -1, \psi(1) = -e. \quad (39)$$

The results of solution of the system (39) using F.G.S and S.G.S iterative methods are given in table 6.

Table 5. *The solution of Ex. (2) as system of second order differential equations using F.G.S and S.G.S methods, h=0.2.*

x	y_{ext}	**F.G.S (39) iterations**	**S.G.S**		$s_0 =$
			$x^{(1)} = 0$	$x^{(1)} = A_1^{-1} y_0$	$x^{(1)} = \frac{2}{2-(\beta+\alpha)} * A_1^{-1} y_0$
			(26)	**(25)**	**(24) iteration**
0.2	1.22140	1.22177	1.22177		
0.4	1.49182	1.49241	1.49241		
0.6	1.82212	1.82275	1.82275		
0.8	2.22554	2.22599	2.22599		

Table 6. *The convergence ratio using F.G.S and S.G.S method for, fourth order, second degree system differential form and integral form for Ex. (2).*

Step Size	**4th order D. E.**		**2nd order D. E. system**			**Integ. Form**		
	R'	**R**	**R'**	**R**	**R/ R'**	**R'**	**R**	**R/ R'**
0.2	Divergent	Divergent	0.163	0.281	**1.7**	3.46	3.6	**1.04**
0.1	Divergent	Divergent	0.039	0.074	**1.8**	3.06	3.24	**1.056**

Table 7. *Shows the convergence ratio between integral form and the system of second degree differential equations for Ex. (2) using F.G.S method and S.G.S method.*

Step Size	**F. G. S**			**S.G.S**		
	$R_{sys.}$	$R_{integ.}$	**Con. Ratio=** $R_{integ.}/R_{sys.}$	$R_{sys.}$	$R_{integ.}$	**Con. Ratio=** $R_{integ.}/R_{sys.}$
0.2	0.163	3.46	21.2	0.281	3.6	12.8
0.1	0.039	3.06	78.2	0.074	3.24	42.2

(ii) when $h = 0.1$ and the finite difference method equation (30) becomes

$$-y_{i-1} + 2y_i - y_{i+1} - 0.01\psi_i = 0, y(0) = 1, y(1) = e, -\psi_{i-1} + 2\psi_i - \psi_{i+1} - 0.01y_i = 0, \psi(0) = -1, \psi(1) = -e. \quad (40)$$

Putting $i = 1(1)9$ we obtain a linear system of algebraic equations, with dimension (18).

6. Conclusion

There is no doubt that for solving large systems the use of iterative techniques is the most appropriate choice. Comparison of application of both the second degree iterative methods and the first degree iterative to the linear systems arising from the two-point BVP and its Fredholm integral form has illustrated the following points.

a Although the three-part splitting sequence does not contains any new parameters than the two-part sequence, the rate of convergence of S.G.S is greater than the rate of convergence F.G.S method especially when we take a suitable choice for the initial vector $x^{(1)}$.

b The spectral radius of the iteration matrix for second degree methods $\rho(B_2)$ in each system is smaller than $\rho(B)$, the spectral radius of F.G.S methods, which makes the rate of convergence in S.G.S method (R) greater than the rate of convergence in F.G.S (R')(note : R/ R'$\cong$ 1.8) see tables 2 and table 3, and hence the number of iteration in S.G.S is less than the number of iterations in F.G.S. especially in linear system arising from BVP form.

c The correct choice of the initial vector $x^{(1)}$decreases the number of iterations see tables 4, 5&6.

d The study is more interesting and more effective in two-point BVP of fourth order since, we found that solving the problem (28) as differential equation the solution is divergent, so we transform it into system of two second order differential equations. Although the dimension of the system reduced from the system of second order differential equation (30) is double the dimension of the system arising from the Fredholm integral equation (31) we found that the number of iterations for (31) is smaller than the number of iterations for (30) see tables 5, 6&7.

e We take different step size in example (2) we found that the rate of convergence R is greater than the rate of convergence R' and R increase rapidly than R' with changing the step size, see table 2 and figures 1& 2.

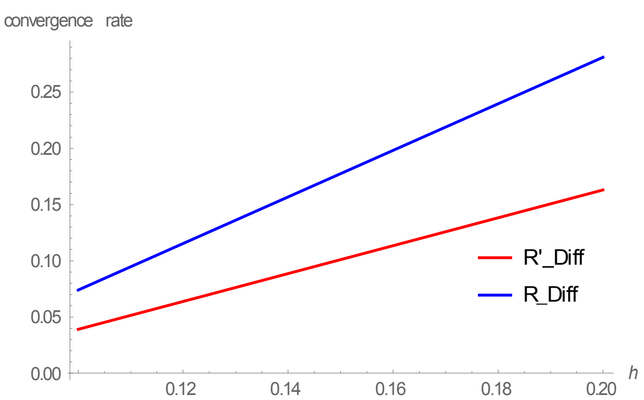

Figure 2. *Shows the priority of S.G.S than F.G.S in second degree system form of Ex. (2).*

f Generally, the results show there is no doubt that solving the problem as integral equation is better than solving it as differential equation in each F.G.S and S.G.S method; especially in S.G.S see tables 3&8.

g Finally, the S.G.S iterative techniques improve the convergence of the linear systems.

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