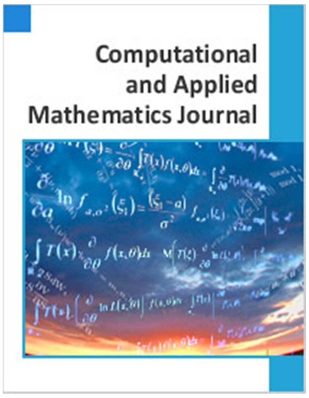
Computational and Applied Mathematics Journal
2017; 3(5): 32-46
http://www.aascit.org/journal/camj
ISSN: 2381-1218 (Print); ISSN: 2381-1226 (Online)

American Association for Science and Technology



Keywords
Batch Arrival,
Service Interruption,
Bernoulli Vacation,
Setup Time,
Transient Solution,
Average Queue Size,
Average Waiting Time

Received: Spril 21, 2017
Accepted: September 7, 2017
Published: September 26, 2017

Three Stage Heterogeneous Service with Feedback Queue and Service Interruption, Setup Time with Bernoulli Vacation

Govindan Ayyappan[1], Pakiradhan Thamizhselvi[1], Karuppannan Sathiya[2]

[1]Department of Mathematics, Pondicherry Engineering College, Puducherry, India
[2]Department of Mathematics, Rajiv Gandhi Govt Arts College, Puducherry, India

Email address
ayyappan@pec.edu (G. Ayyappan), tmlselvi972@gmail.com (P. Thamizhselvi), sathiyathiyagumaths@gmail.com (K. Sathiya)

Citation
Govindan Ayyappan, Pakiradhan Thamizhselvi, Karuppannan Sathiya. Three Stage Heterogeneous Service with Feedback Queue and Service Interruption, Setup Time with Bernoulli Vacation. *Computational and Applied Mathematics Journal.* Vol. 3, No. 5, 2017, pp. 32-46.

Abstract
This paper deals with the study of batch arrival queue with a single server providing three stages of heterogeneous service, subject to random interruption and vacation. As soon as the completion of third stage service, if the customer is dissatisfied with its service, he can immediately join the tail of the original queue as a feedback customer. After completion of the three stages of service in succession to each customer the server has the option to take a vacation of random length with probability θ or to continue staying in system with probability (1-θ). While serving the customer, we assume interruptions arrive at random according to a Poisson process with mean rate $\alpha$and βbe the rate of attending interruption. Before providing service to a new customer or a batch of customers that joins the system in the renewed busy period, the server enters into a random setup time process. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results are obtained explicitly. Also the average number of customer in the queue and the average waiting time are derived. Numerical results are computed.

1. Introduction

The study on queuing models have become an indispensable area due to its wide applicability in real life situations like computer networks, telecommunication networks, health sectors, manufacturing and production sections etc.,

Vacatin queues have been studied by several authors including Doshi (1986), Takagi (1990), Chae et al. (2001). Madan. K. C et al. (2005) andThangaraj. V and Vanitha. S (2010a), have studied Non-Morkovian feedback queueing model with two types of service and optional server vacations. Thangaraj. V and Vanitha. S (2010b) and Maragatha Sundari. S and Srinivasan. S (2012) have studied a single server with compulsory server vacation and service interruptions. Maragatha Sundari, S. and Karthikeyan, K. (2015a, 2015b) have studied a batch arrival of two stages with standby server during general vacation time & general repair time and a study on M/G/1 queueing system with extended vacation, random breakdowns and general repair, Madan. K. C. et al. (2011) studied batch arrival queue with Bernoulli schedule general vacation times, general extended vacations, random breakdowns, general delay times for repairs to start and general repair times.

Choudhury. G (2000, 2008), have studied a batch arrival queue with setup period and server vacations. Ayyappan. G et al (2014) have studied a batch arrival queue with setup time, Bernoulli vacation, Breakdown and Delayed Repair.

This paper consider a $M^{[X]}/G/1$ feedback queue with three stage service, set up time, server interruption and vacations. Each customer undergoes three stage of heterogeneous service with general (arbitrary) service time distributions. As soon as the completion of third stage service, if the customer is dissatified with its service, he can immediately join the tail of the original queue as a feedback customer for receiving service with probability p. Otherwise the customer may depart forever from the system with probability $(1-p)$. After completion of the three stages of service in succession to each customer, the server has the option to take a vacation of random length with probability θ or to continue staying in system with probability $(1-\theta)$. If the server is ready for service in the system, then the system becomes operative only when a new customer or a batch of customers arrives to the system. The server startup corresponds to the preparatory work of the server before starting the service. In some actual situations, the server often needs a startup time before providing service. In this case, it will take a random setup time before it actually starts serving a new customer. This random setup time is usually termed as SET (during which no proper work is done) in order to set the system into operative mode before actual service begins (setup period). On the account of that, the system may be subject to breakdowns; the breakdowns occur according to Poisson process. Once the system breakdowns, the repair process will be started immediately which follows exponential distribution. After the repair process is completed, the server resumes its work immediately. Also, whenever the system meets a breakdown, the customer whose service is interrupted goes back to the head of the queue and the interrupted customer restarts its service from the beginning again.

Here we derive the time dependent probability generating functions in terms of laplace transforms. Also derive the average queue size, system size and average waiting time in the queue and the system. Some particular cases and numerical results with graphical illustration are also discussed.

The rest of the paper is organized as follows. Mathematical description of our model in section (2). Definitions, equations governing of our model and the time dependent solution have been obtained in section (3). The corresponding steady state results have been derived explicitly in section (4). Average queue size and the average waiting time are computed in Section (5). Some particular cases have been discussed in section (6). In Section (7), we consider a numerical results of our model.

2. Mathematical Description of Our Model

1. Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i dt (i=1,2,3,...)$ be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt)$, where $0 \le c_i \le 1, \sum_{i=1}^{\infty} c_i = 1$, and $\lambda > 0$ are the average arrival rates of batches. The customers are served one-by-one on a "first come-first served basis" basis.

2. The random setup time is a random variable called SET variable following exponential distribution with mean setup time being v.

3. Each customer undergoes three stages of heterogeneous service provided by a single server on a first come- first served basis. The service time of the three stages follows different general (arbitrary) distributions with distribution function $B_i(v)$ and the density function $b_i(v)(i=1,2,3)$.

4. Let $\mu_i(x)dx$ be the conditional probability of completion of the ith stage of service during the interval $(x, x+dx]$, given that the elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}, i=1,2,3$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, i=1,2,3.$$

5. After completion of three stages of service, if the customer is dissatisfied with its service for certain reasons, the customer may immediately join the tail of the original queue as a feedback customer for receiving service with probability p $(0 \le p \le 1)$. Otherwise the customer may depart forever from the system with probability $(1-p)$. The service discipline for feedback and newly customers are first come first served. Also service time for a feedback customer is independent of its previous service times.

6. As soon as the customer's third stage service is completed, the server may go for a vacation of random length V with probability $\theta (0 \le \theta \le 1)$ or it may continue to serve the next customer with probability $(1-\theta)$.

7. The vacation time follow general (arbitrary) distribution with distribution function $V(s)$ and the density function $v(s)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1-V(x)}$$

and therefore,

$$v(s) = \gamma(s) e^{-\int_0^s \gamma(x)dx}$$

8. On returning from vacation, the server instantly starts serving the customer at the head of the queue, if any. The server stays for being available if there are no customers.

9. The system may break down at random and breakdowns are assumed to occur according to a Poisson stream with mean breakdown rate $\alpha > 0$.

10. Once the system breaks down, it enters a repair process immediately. The repair times are exponentially distributed with mean $\frac{1}{\beta}$.

11. Various stochastic processes involved in the system are assumed to be independent of each other.

3. Definitions and Equations Governing the System

Let us define

1. $S_n(t) =$ Probability that at time t, the server is in setup time while there are n $(n \geq 1)$ customers in the queue.

2. $P_n^{(i)}(x,t) =$ Probability that at time t, the server is active providing ith stage of service and there are $n\,(n \geq 0)$ customers in the queue excluding the one being served and the elapsed service time for this customer is x. Accordingly, $P_n^{(i)}(t) = \int_0^\infty P_n^{(i)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the ith stage service irrespective of the value of x, where $i = 1,2,3.$

3. $V_n(x,t) =$ Probability that at time t, the server is on vacation with elapsed vacation time x, and there are $n\,(n \geq 0)$ customers in the queue. Accordingly, $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under vacation irrespective of the value of x.

4. $R_n(t) =$ Probability that at time t, the server is inactive due to breakdown and the system is under repair, while there are $n\,(n \geq 1)$ customers in the queue.

5. $Q(t) =$ Probability that at time t, there are no customer in the system and the server is idle but available in the system.

$$\frac{d}{dt}S_n(t) = -(\lambda + \nu)S_n + \lambda\sum_{i=1}^{n-1} c_i S_{n-i}(t) + \lambda C_n Q(t), n \geq 1 \tag{1}$$

$$\frac{\partial}{\partial t}P_0^{(1)}(x,t) + \frac{\partial}{\partial x}P_0^{(1)}(x,t) = -(\lambda + \mu_1(x) + \alpha)P_0^{(1)}(x,t) \tag{2}$$

$$\frac{\partial}{\partial t}P_n^{(1)}(x,t) + \frac{\partial}{\partial x}P_n^{(1)}(x,t) = -(\lambda + \mu_1(x) + \alpha)P_n^{(1)}(x,t) + \lambda\sum_{i=1}^{n} C_i P_{n-i}^{(1)}(x,t),\ n \geq 1, \tag{3}$$

$$\frac{\partial}{\partial t}P_0^{(2)}(x,t) + \frac{\partial}{\partial x}P_0^{(2)}(x,t) = -(\lambda + \mu_2(x) + \alpha)P_0^{(2)}(x,t) \tag{4}$$

$$\frac{\partial}{\partial t}P_n^{(2)}(x,t) + \frac{\partial}{\partial x}P_n^{(2)}(x,t) = -(\lambda + \mu_2(x) + \alpha)P_n^{(2)}(x,t) + \lambda\sum_{i=1}^{n} C_i P_{n-i}^{(2)}(x,t),\ n \geq 1, \tag{5}$$

$$\frac{\partial}{\partial t}P_0^{(3)}(x,t) + \frac{\partial}{\partial x}P_0^{(3)}(x,t) = -(\lambda + \mu_3(x) + \alpha)P_0^{(3)}(x,t) \tag{6}$$

$$\frac{\partial}{\partial t}P_n^{(3)}(x,t) + \frac{\partial}{\partial x}P_n^{(3)}(x,t) = -(\lambda + \mu_3(x) + \alpha)P_n^{(3)}(x,t) + \lambda\sum_{i=1}^{n} C_i P_{n-i}^{(3)}(x,t),\ n \geq 1, \tag{7}$$

$$\frac{\partial}{\partial t}V_0(x,t) + \frac{\partial}{\partial x}V_0(x,t) = -(\lambda + \gamma(x))V_0(x,t) \tag{8}$$

$$\frac{\partial}{\partial t}V_n(x,t) + \frac{\partial}{\partial x}V_n(x,t) = -(\lambda + \gamma(x))V_n(x,t) + \lambda\sum_{i=1}^{n} C_i V_{n-i}(x,t), n \geq 1, \tag{9}$$

$$\frac{d}{dt}R_n(t) = -(\lambda + \beta)R_n(t) + \lambda\sum_{i=1}^{n-1} c_i R_{n-i}(t) + \alpha\int_0^\infty P_{n-1}^{(1)}(x,t)dx + \alpha\int_0^\infty P_{n-1}^{(2)}(x,t)dx + \alpha\int_0^\infty P_{n-1}^{(3)}(x,t)dx, n \geq 1 \tag{10}$$

$$\frac{d}{dt}Q(t)=-\lambda Q(t)+\int_0^\infty V_0(x,t)\gamma(x)dx+(1-\theta)(1-p)\int_0^\infty P_0^{(3)}(x,t)\mu_3(x)dx \tag{11}$$

The above set of equations are to be solved under the following boundary conditions at $x=0$.

$$P_n^{(1)}(0,t)=\beta R_{n+1}(t)+\int_0^\infty V_{n+1}(x,t)\gamma(x)dx+(1-\theta)p\int_0^\infty P_n^{(3)}(x,t)\mu_3(x)dx+\nu S_{n+1}(t), n\geq 0 \tag{12}$$

$$P_n^{(2)}(0,t)=\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx, n\geq 0 \tag{13}$$

$$P_n^{(3)}(0,t)=\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx, n\geq 0 \tag{14}$$

$$V_n(0,t)=\theta\left[(1-p)\int_0^\infty P_n^{(3)}(x,t)\mu_3(x)dx+p\int_0^\infty P_{n-1}^{(3)}(x,t)\mu_3(x)dx\right], \quad n\geq 0 \tag{15}$$

assume that initially there are no customer in the system and the server is idle.
So the initial conditions are

$$P_n^j(0)=0, \, for\ j=1,2,3., n=0,1,2,..., Q(0)=1$$

$$and\ V_0(0)=V_n(0)=0, R_0(0)=R_n(0)=0\ for\ n=1,2,3,... \tag{16}$$

Next, we define the following probability generating functions:

$$\left\{\begin{aligned} &P^j(x,z,t)=\sum_{n=0}^{\infty}z^nP_n^j(x,t), P^j(z,t)=\sum_{n=0}^{\infty}z^nP_n^j(t),\\ &V(x,z,t)=\sum_{n=0}^{\infty}z^nV_n(x,t), V(z,t)=\sum_{n=0}^{\infty}z^nV_n(t),\\ &S(z,t)=\sum_{n=1}^{\infty}z^nS_n(t), R(z,t)=\sum_{n=1}^{\infty}z^nR_n(t),\\ &C(z)=\sum_{n=1}^{\infty}z^nc_n. \end{aligned}\right\} \tag{17}$$

which are convergent inside the circle given by $|z|\leq 1$, and define the Laplace transform of afunction $f(t)$ as

$$\overline{f}(s)=\int_0^\infty f(t)e^{-st}dt.$$

$$(s+\lambda+\nu)\overline{S}_n(s)=\lambda\sum_{i=1}^{n-1}c_i\overline{S}_{n-i}(s)+\lambda c_n\overline{Q}(s), n\geq 1 \tag{18}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(1)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\overline{P}_0^{(1)}(x,s)=0 \tag{19}$$

$$\frac{\partial}{\partial x}\overline{P}_n^{(1)}(x,s)+(s+\lambda+\mu_1(x)+\alpha)\overline{P}_n^{(1)}(x,s)=\lambda\sum_{i=1}^{n}c_i\overline{P}_{n-i}^{(1)}(x,s), n\geq 1, \tag{20}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(2)}(x,s)+(s+\lambda+\mu_2(x)+\alpha)\overline{P}_0^{(2)}(x,s)=0 \tag{21}$$

$$\frac{\partial}{\partial x}\overline{P}_n^{(2)}(x,s)+(s+\lambda+\mu_2(x)+\alpha)\overline{P}_n^{(2)}(x,s)=\lambda\sum_{i=1}^{n}c_i\overline{P}_{n-i}^{(2)}(x,s), n\geq 1, \tag{22}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(3)}(x,s)+(s+\lambda+\mu_3(x)+\alpha)\overline{P}_0^{(3)}(x,s)=0 \tag{23}$$

$$\frac{\partial}{\partial x}\overline{P}_n^{(3)}(x,s)+(s+\lambda+\mu_3(x)+\alpha)\overline{P}_n^{(3)}(x,s)=\lambda\sum_{i=1}^{n}c_i\overline{P}_{n-i}^{(3)}(x,s), \qquad n\geq 1 \tag{24}$$

$$\frac{\partial}{\partial x}\overline{V}_0(x,s)+(s+\lambda+\gamma(x))\overline{V}_0(x,s)=0 \tag{25}$$

$$\frac{\partial}{\partial x}\overline{V}_n(x,s)+(s+\lambda+\gamma(x))\overline{V}_n(x,s)=\lambda\sum_{i=1}^{n}c_i\overline{V}_{n-i}(x,s), \qquad n\geq 1 \tag{26}$$

$$(s+\lambda+\beta)\overline{R}_n(s)=\lambda\sum_{i=1}^{n-1}c_i\overline{R}_{n-i}(s)+\alpha\int_0^\infty\overline{P}_{n-1}^{(1)}(x,s)dx+\alpha\int_0^\infty\overline{P}_{n-1}^{(2)}(x,s)dx+\alpha\int_0^\infty\overline{P}_{n-1}^{(3)}(x,s)dx, n\geq 1 \tag{27}$$

$$(s+\lambda)\overline{Q}(s)-1=\int_0^\infty\overline{V}_0(x,s)\gamma(x)dx+(1-\theta)(1-p)\int_0^\infty\overline{P}_0^{(3)}(x,s)\mu_3(x)dx \tag{28}$$

$$\overline{P}_n^{(1)}(0,s)=\beta\overline{R}_{n+1}(s)+\int_0^\infty\overline{V}_{n+1}(x,s)\gamma(x)dx+(1-\theta)(1-p)\int_0^\infty\overline{P}_{n+1}^{(3)}(x,s)\mu_3(x)dx+(1-\theta)p\int_0^\infty\overline{P}_n^{(3)}(x,s)\mu_3(x)dx+\nu\overline{S}_{n+1}(s), n\geq 0 \tag{29}$$

$$\overline{P}_n^{(2)}(0,s)=\int_0^\infty\overline{P}_n^{(1)}(x,s)\mu_1(x)dx, n\geq 0 \tag{30}$$

$$\overline{P}_n^{(3)}(0,s)=\int_0^\infty\overline{P}_n^{(2)}(x,s)\mu_2(x)dx, n\geq 0 \tag{31}$$

$$\overline{V}_n(0,s)=\theta\left[(1-p)\int_0^\infty\overline{P}_n^{(3)}(x,s)\mu_3(x)dx+p\int_0^\infty\overline{P}_{n-1}^{(3)}(x,s)\mu_3(x)dx\right], \qquad n\geq 0 \tag{32}$$

Next, multiplythe equations (18) and (27) by z^n and summing over n from 1 to ∞, andequations $(20),(22),(24)$ and (26) by appropriate powers of z^n and summing over n from1 to ∞, adding these to equations $(19),(21),(23)$ and (25) using equations (16) and (17), thus we get on simplification

$$\overline{S}(z,s)=\frac{\lambda C(z)\overline{Q}(s)}{(s+\lambda-\lambda C(z)+\nu)} \tag{33}$$

$$\frac{\partial}{\partial x}\overline{P}^{(1)}(x,z,s)+(s+\lambda-\lambda C(z)+\alpha+\mu_1(x))\overline{P}^{(1)}(x,z,s)=0 \tag{34}$$

$$\frac{\partial}{\partial x}\overline{P}^{(2)}(x,z,s)+(s+\lambda-\lambda C(z)+\alpha+\mu_2(x))\overline{P}^{(2)}(x,z,s)=0 \tag{35}$$

$$\frac{\partial}{\partial x}\overline{P}^{(3)}(x,z,s)+(s+\lambda-\lambda C(z)+\alpha+\mu_3(x))\overline{P}^{(3)}(x,z,s)=0 \tag{36}$$

$$\frac{\partial}{\partial x}\overline{V}(x,z,s)+(s+\lambda-\lambda C(z)+\gamma(x))\overline{V}(x,z,s)=0 \tag{37}$$

$$(s+\lambda-\lambda C(z)+\beta)\overline{R}(z,s)=\alpha z\int_0^\infty\overline{P}^{(1)}(x,z,s)dx+\alpha z\int_0^\infty\overline{P}^{(2)}(x,z,s)dx+\alpha z\int_0^\infty\overline{P}^{(3)}(x,z,s)dx \tag{38}$$

Now multiply the equation (29) by z^n summing over n from 0 to ∞, we have

$$z\overline{P}^{(1)}(0,z,s)=\beta\overline{R}(z,s)+\nu\overline{S}(z,s)+\int_0^\infty\overline{V}(x,z,s)\gamma(x)dx+(1-\theta)(1-p)\int_0^\infty\overline{P}^{(3)}(x,z,s)\mu_3(x)dx$$

$$+(1-\theta)pz\int_0^\infty\overline{P}^{(3)}(x,z,s)\mu_3(x)dx-[\int_0^\infty\overline{V}_0(x,s)\gamma(x)dx+(1-\theta)(1-p)\int_0^\infty\overline{P}_0^{(3)}(x,s)\mu_3(x)dx] \quad (39)$$

use equation (28) in equation (39) we get

$$z\overline{P}^{(1)}(0,z,s)=\beta\overline{R}(z,s)+\nu\overline{S}(z,s)+\int_0^\infty\overline{V}(x,z,s)\gamma(x)dx+(1-\theta)(1-p)\int_0^\infty\overline{P}^{(3)}(x,z,s)\mu_3(x)dx$$

$$+(1-\theta)pz\int_0^\infty\overline{P}^{(3)}(x,z,s)\mu_3(x)dx-[(s+\lambda)\overline{Q}(s)-1] \quad (40)$$

performing similar operations in equations $(30),(31)$ and equation (32), we get

$$\overline{P}_n^{(2)}(0,z,s)=\int_0^\infty\overline{P}^{(1)}(x,z,s)\mu_1(x)dx,\ n\geq 0 \quad (41)$$

$$\overline{P}_n^{(3)}(0,z,s)=\int_0^\infty\overline{P}^{(2)}(x,z,s)\mu_2(x)dx,\ n\geq 0 \quad (42)$$

$$\overline{V}(0,z,s)=\theta(1-p)\int_0^\infty\overline{P}^{(3)}(x,z,s)\mu_3(x)dx+\theta p\int_0^\infty\overline{P}^{(3)}(x,z,s)\mu_3(x)dx,\ n\geq 0 \quad (43)$$

Now integrate equation (34) between 0 to x, we get

$$\overline{P}^{(1)}(x,z,s)=\overline{P}^{(1)}(0,z,s)e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x\mu_1(t)dt} \quad (44)$$

where $\overline{P}^{(1)}(0,z,s)$ is given by equation (40).

Again integrating equation (44) by parts with respect to x, we have

$$\overline{P}^{(1)}(z,s)=\overline{P}^{(1)}(0,z,s)\left[\frac{1-\overline{B}_1(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right] \quad (45)$$

$$\overline{B}_1(s+\lambda-\lambda C(z)+\alpha)=\int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}dB_1(x) \quad (46)$$

is the Laplace-Stieltjes transform of the first stage service time $B_1(x)$.

Now, we multiplying both sides of equation (44) by $\mu_1(x)$ and integrating over x, we get

$$\int_0^\infty\overline{P}^{(1)}(x,z,s)\mu_1(x)dx=\overline{P}^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda C(z)+\alpha) \quad (47)$$

Similarly on integrating equations (35) to (37) from 0 to x, we get

$$\overline{P}^{(2)}(x,z,s)=\overline{P}^{(2)}(0,z,s)e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x\mu_2(t)dt} \quad (48)$$

$$\overline{P}^{(3)}(x,z,s)=\overline{P}^{(3)}(0,z,s)e^{-(s+\lambda-\lambda C(z)+\alpha)x-\int_0^x\mu_3(t)dt} \quad (49)$$

$$\overline{V}(x,z,s)=\overline{V}(0,z,s)e^{-(s+\lambda-\lambda C(z))x-\int_0^x \gamma(t)dt} \tag{50}$$

where $\overline{P}^{(2)}(0,z,s),\overline{P}^{(3)}(0,z,s)$ and $\overline{V}(0,z,s)$ is given by equations $(41),(42)$ and (43).

Again integrating equations $(48),(49)$ and (50) with respect to x we get

$$\overline{P}^{(2)}(z,s)=\overline{P}^{(2)}(0,z,s)\left[\frac{1-\overline{B}_2(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right] \tag{51}$$

$$\overline{P}^{(3)}(z,s)=\overline{P}^{(3)}(0,z,s)\left[\frac{1-\overline{B}_3(s+\lambda-\lambda C(z)+\alpha)}{(s+\lambda-\lambda C(z)+\alpha)}\right] \tag{52}$$

$$\overline{V}(z,s)=\overline{V}(0,z,s)\left[\frac{1-\overline{V}(s+\lambda-\lambda C(z))}{(s+\lambda-\lambda C(z))}\right] \tag{53}$$

where

$$\overline{B}_2(s+\lambda-\lambda C(z)+\alpha)=\int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}dB_2(x) \tag{54}$$

$$\overline{B}_3(s+\lambda-\lambda C(z)+\alpha)=\int_0^\infty e^{-(s+\lambda-\lambda C(z)+\alpha)x}dB_3(x) \tag{55}$$

$$\overline{V}(s+\lambda-\lambda C(z))=\int_0^\infty e^{-(s+\lambda-\lambda C(z))x-\int_0^x \gamma(t)dt}dx \tag{56}$$

are the Laplace-Stieltjes transform of the second stage service time, third stage service timeand vacation time $B_2(x)$, $B_3(x)$ and $V(x)$ respectively.

Now, we multiplying both sides of equation (48) by $\mu_2(x)$, (49) by $\mu_3(x)$ and (50) by $\gamma(x)$ and integrating over x, we obtain

$$\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)dx=\overline{P}^{(2)}(0,z,s)\overline{B}_2(s+\lambda-\lambda C(z)+\alpha) \tag{57}$$

$$\int_0^\infty \overline{P}^{(3)}(x,z,s)\mu_3(x)dx=\overline{P}^{(3)}(0,z,s)\overline{B}_3(s+\lambda-\lambda C(z)+\alpha) \tag{58}$$

$$\int_0^\infty \overline{V}(x,z,s)\gamma(x)dx=\overline{V}(0,z,s)\overline{V}(s+\lambda-\lambda C(z)) \tag{59}$$

Substitute equation (47) in equation (41)

$$\overline{P}^{(2)}(0,z,s)=\overline{P}^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda C(z)+\alpha) \tag{60}$$

Substitute equation (57) in equation (42) and use equation (60)

$$\overline{P}^{(3)}(0,z,s)=\overline{P}^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda C(z)+\alpha)\times\overline{B}_2(s+\lambda-\lambda C(z)+\alpha) \tag{61}$$

Substitute equation (58) in equation (43) and use equation (61)

$$\overline{V}(0,z,s)=\theta[(1-p+pz)]\overline{P}^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda C(z)+\alpha)\times\overline{B}_2(s+\lambda-\lambda C(z)+\alpha)\times\overline{B}_3(s+\lambda-\lambda C(z)+\alpha). \tag{62}$$

Using equations $(44),(48)$ and (49) in equation (38), we obtain

$$\overline{R}(z,s)=\left[\frac{\alpha z\overline{P}^{(1)}(0,z,s)}{(s+\lambda-\lambda C(z)+\alpha)(s+\lambda-\lambda C(z)+\beta)}\right]\times\left[1-\overline{B}(s+\lambda-\lambda C(z)+\alpha)\right] \tag{63}$$

where

$$\overline{B}_1(a)\overline{B}_2(a)\overline{B}_3(a)=\overline{B}(a)$$

Now, substituting equations $(33),(58),(59)$ and (63) in equation (40) gives

$$\overline{P}^{(1)}(0,z,s)=\left[\frac{ab\left[(1-S\overline{Q}(s))+\lambda(\frac{\nu C(z)}{d}-1)\overline{Q}(s)\right]}{Dr}\right] \tag{64}$$

where

$$a=(s+\lambda-\lambda C(z)+\alpha), b=(s+\lambda-\lambda C(z)+\beta), c=(s+\lambda-\lambda C(z)), d=(s+\lambda-\lambda C(z)+\nu)$$

and

$$Dr=ab\Big(z-(1-P+Pz)\overline{B}(a)[\theta\overline{V}(c)+1-\theta]\Big)-\alpha\beta z[1-\overline{B}(a)] \tag{65}$$

By substitute equation (64) in equations (45), (51), (52) and (53), we get

$$\overline{P}^{(1)}(z,s)=\left(\frac{b\left[(1-S\overline{Q}(s))+\lambda(\frac{\nu C(z)}{d}-1)\overline{Q}(s)\right]\left[1-\overline{B}_1(a)\right]}{Dr}\right) \tag{66}$$

$$\overline{P}^{(2)}(z,s)=\left(\frac{b\left[(1-S\overline{Q}(s))+\lambda(\frac{\nu C(z)}{d}-1)\overline{Q}(s)\right]\overline{B}_1(a)\left[1-\overline{B}_2(a)\right]}{Dr}\right) \tag{67}$$

$$\overline{P}^{(3)}(z,s)=\left(\frac{b\left[(1-S\overline{Q}(s))+\lambda(\frac{\nu C(z)}{d}-1)\overline{Q}(s)\right]\overline{B}_1(a)\overline{B}_2(a)\left[1-\overline{B}_3(a)\right]}{Dr}\right) \tag{68}$$

$$\overline{V}(z,s)=\theta(1-P+Pz)ab\overline{B}(a)\times\left(\frac{\left[(1-S\overline{Q}(s))+\lambda(\frac{\nu C(z)}{d}-1)\overline{Q}(s)\right]\left[1-\overline{V}(c)\right]}{cDr}\right) \tag{69}$$

From equation (33)

$$\overline{S}(z,s)=\frac{\lambda C(z)\overline{Q}(s)}{d} \tag{70}$$

Substitute equation (64) in equation (63)

$$\overline{R}(z,s)=\alpha z\left[\frac{\left[(1-S\overline{Q}(s))+\lambda(\frac{\nu C(z)}{d}-1)\overline{Q}(s)\right]\left[1-\overline{B}(a)\right]}{Dr}\right] \tag{71}$$

where Dr is given by equation (65). Thus $\overline{P}^{(1)}(z,s),\overline{P}^{(2)}(z,s),\overline{P}^{(3)}(z,s),\overline{V}(z,s),\overline{S}(z,s)$ and $\overline{R}(z,s)$ are completely determined from equations (66) to (71).

4. Steady State Results

In this section, we derive the steady state probability distribution for our queueing model.
By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s)=\lim_{t\to\infty} f(t).$$

In order to determine Q, we use the normalizing condition

$$S(1)+P^1(1)+P^2(1)+P^3(1)+V(1)+R(1)+Q=1.$$

The steady state probability for an $M^{[X]}/G/1$ queue with three stage service with feedback queue, random breakdown and repair, Bernoulli vacation and setup time are given by

$$P^{(1)}(z)=\left(\frac{f_2(z)\lambda Q(\nu+\lambda)(C(z)-1)\left[1-\overline{B}_1(f_1(z))\right]}{f_3(z)dr}\right) \tag{72}$$

$$P^{(2)}(z)=\left(\frac{f_2(z)\lambda Q(\nu+\lambda)(C(z)-1)\overline{B}_1(f_1(z))\left[1-\overline{B}_2(f_1(z))\right]}{f_3(z)dr}\right) \tag{73}$$

$$P^{(3)}(z)=\left(\frac{f_2(z)\lambda Q(\nu+\lambda)(C(z)-1)\overline{B}_1(f_1(z))\overline{B}_2(f_1(z))\left[1-\overline{B}_3(f_1(z))\right]}{f_3(z)dr}\right) \tag{74}$$

$$V(z)=\left(\frac{f_1(z)f_2(z)Q\theta(1-P+Pz)(\nu+\lambda)\overline{B}(f_1(z))\left[\overline{V}(f_4(z))-1\right]}{f_3(z)dr}\right) \tag{75}$$

$$R(z)=\left(\frac{\lambda z(C(z)-1)\alpha Q(\nu+\lambda)\left[1-\overline{B}(f_1(z))\right]}{f_3(z)dr}\right) \tag{76}$$

where

$$f_1(z)=(\lambda-\lambda C(z)+\alpha), f_2(z)=(\lambda-\lambda C(z)+\beta), f_3(z)=(\lambda-\lambda C(z)+\nu), f_4(z)=(\lambda-\lambda C(z)),$$

$$\text{and } dr=f_1(z)f_2(z)\Big(z-(1-P+Pz)\overline{B}(f_1(z))\left[\theta\overline{V}(f_4(z))+1-\theta\right]\Big)-\alpha\beta z\left[1-\overline{B}(f_1(z))\right]. \tag{77}$$

Let $W_q(z)$ be the probability generating function of the queue size irrespective of the state of the system. Then adding equations (72) to (76) we obtain

$$W_q(z)=P^{(1)}(z)+P^{(2)}(z)+P^{(3)}(z)+V(z)+R(z)$$

$$W_q(z) = \left(\frac{1}{[f_3(z)dr]}\right)[(\nu+\lambda)Q\lambda(C(z)-1)\left[1-\overline{B}_1(f_1(z))\right]\left[f_2(z)+\alpha z\right]$$

$$+f_1(z)f_2(z)\theta(\nu+\lambda)Q(1-P+Pz)\overline{B}(f_1(z))\left[\overline{V}(f_4(z))-1\right]] \quad (78)$$

where dr is given by equation (77).

We see that for $z=1, W_q(z)$ is indeterminate of the $0/0$ form. Therefore, we apply L'Hopital'srule and on simplification we obtain the result of equation (78) where $C(1)=1, C'(1)=E(I)$ is average batch size of the arriving customers, $\overline{V}'(0)=-E(V)$ the average vacation time.

$$W_q(1) = \left(\frac{\lambda QE(I)(\nu+\lambda)\left((\alpha+\beta)\left[1-\overline{B}(\alpha)\right]+\alpha\beta\theta\overline{B}(\alpha)E(V)\right)}{\nu\left(\alpha\beta\overline{B}(\alpha)\left[(1-P)-\theta E(\nu)\lambda E(I)\right)-\lambda E(I)(\alpha+\beta)(1-\overline{B}(\alpha))\right)}\right) \quad (79)$$

$$S(1) = \frac{\lambda Q}{\nu}. \quad (80)$$

Then $W_q(1)+S(1)+Q=1$, we get

$$Q = 1-\rho \quad (81)$$

$$Q = \frac{1}{(1+\frac{\lambda}{\nu})}[1-\lambda E(I)(\frac{1}{\beta\overline{B}(\alpha)(1-P)}-\frac{1}{\beta(1-P)}+\frac{1}{\alpha\overline{B}(\alpha)(1-P)}-\frac{1}{\alpha(1-P)}+\frac{\theta E(V)}{(1-P)})] \quad (82)$$

the utilization factor of the system is given by

$$\rho = \frac{1}{(1+\frac{\lambda}{\nu})}[\frac{\lambda}{\nu}+\lambda E(I)(\frac{1}{\beta\overline{B}(\alpha)(1-P)}-\frac{1}{\beta(1-P)}+\frac{1}{\alpha\overline{B}(\alpha)(1-P)}-\frac{1}{\alpha(1-P)}+\frac{\theta E(V)}{(1-P)})] \quad (83)$$

where $\rho<1$ is the stability condition under which the steady state exists. Equation (82) gives the probability that the server is idle. Substituting equation (82) in equation (78), we have completely and explicitly determined $W_q(z)$, the probability generating function ofthe queue size is

$$W_q(z) = \left(\frac{1}{[f_3(z)dr]}\right)\left(\frac{1}{(1+\frac{\lambda}{\nu})}\right)[1-\lambda E(I)(\frac{1}{\beta\overline{B}(\alpha)(1-P)}-\frac{1}{\beta(1-P)}+\frac{1}{\alpha\overline{B}(\alpha)(1-P)}-\frac{1}{\alpha(1-P)}+\frac{\theta E(V)}{(1-P)})]$$

$$\times[(\nu+\lambda)\lambda(C(z)-1)[1-\overline{B}_1(f_1(z))][f_2(z)+\alpha z]+f_1(z)f_2(z)\theta(\nu+\lambda)(1-P+Pz)\overline{B}(f_1(z))[\overline{V}(f_4(z))-1]]. \quad (84)$$

Again the steady state results for many particular cases can be derived from the result given inequation (84).

5. Average Queue Size and Average Waiting Time

Let L_q denote the mean number of customers in the queue under the steady state. Then

$$L_q = \frac{d}{dz}(W_q(z))|_{z=1} + \frac{d}{dz}S(z)|_{z=1}$$

since this formula $W_q(z)$ gives the indeterminate of the form $0/0$, then we write $W_q(z)$ givenin (78) as $W_q(z)=\frac{N(z)}{D(z)}$

where $N(z)$ and $D(z)$ are numerator and denominator of the righthand side of (78) respectively. Then we use

$$\lim_{z\to 1}\frac{d}{dz}(W_q(z)) = \lim_{z\to 1}\left(\frac{D'(z)N''(z)-N'(z)D''(z)}{2(D'(z))^2}\right) = \left(\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right) \quad (85)$$

where primes and double primes in the equations denote the first and second derivatives of thefunctions. Carrying out the derivatives of the funtcions and evaluating at $z=1$, we have

$$\lim_{z\to 1}\frac{d}{dz}S(z) = \frac{\lambda E(I)Q(\nu+\lambda)}{\nu^2} \quad (86)$$

Therefore

$$L_q = \left(\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right) + \frac{\lambda E(I)Q(\nu+\lambda)}{\nu^2} \quad (87)$$

where

$$N'(1) = \lambda QE(I)(\nu+\lambda)\Big((\alpha+\beta)[1-\overline{B}(\alpha)] + \alpha\beta\theta\overline{B}(\alpha)E(V)\Big) \quad (88)$$

$$N''(1) = (\nu+\lambda)Q[(\lambda E[I(I-1)](\alpha+\beta) + 2\lambda E(I)(-\lambda E(I)+\alpha))(1-\overline{B}(\alpha)) + 2(\lambda E(I))^2(\overline{B'}_1(\alpha)\overline{B}_2(\alpha)\overline{B}_3(\alpha) + \overline{B}_1(\alpha)\overline{B'}_2(\alpha)\overline{B}_3(\alpha)$$
$$+\overline{B}_1(\alpha)\overline{B}_2(\alpha)\overline{B'}_3(\alpha))[\alpha+\beta-\alpha\beta\theta E(V)] + \overline{B}(\alpha)[\alpha\beta((\lambda E(I))^2\theta E(V^2) + \lambda E[I(I-1)]\theta E(V))$$
$$-2(\lambda E(I))^2(\alpha+\beta)\theta E(V) + 2\alpha\beta\theta pE(V)\lambda E(I)] \quad (89)$$

$$D'(1) = \nu(\alpha\beta\overline{B}(\alpha)[(1-p)-\theta E(V)\lambda E(I)] - \lambda E(I)(\alpha+\beta)[1-\overline{B}(\alpha)]) \quad (90)$$

$$D''(1) = \nu[(-\lambda E[I(I-1)](\alpha+\beta) + 2(\lambda E(I))^2)(1-\overline{B}(\alpha)) - 2(\lambda E(I))(\alpha+\beta)[1-p\overline{B}(\alpha) - \overline{B}(\alpha)\theta E(V)\lambda E(I)]$$
$$+\overline{B'}(\alpha)[-2(\lambda E(I))^2(\alpha+\beta) + 2(\lambda E(I))^2\alpha\beta\theta E(V) - 2\alpha\beta(1-p)\lambda E(I)] - \overline{B}(\alpha)[\alpha\beta\theta[E(V^2)(\lambda E(I))^2$$
$$+E(V)\lambda E[I(I-1)]] + 2\alpha\beta\theta pE(V)\lambda E(I)] + \frac{1}{\nu}[-2\lambda E(I)(-\lambda E(I)(\alpha+\beta)(1-\overline{B}(\alpha)) + \alpha\beta\overline{B}(\alpha)[(1-p)-\theta E(V)\lambda E(I)])] \quad (91)$$

$E(V^2)$ is the second moment of vacation time and $E[I(I-1)]$ is the second factorial moment ofthe batch size of arriving customer.

Then we substitute the values $N'(1), N''(1), D'(1), D''(1)$ from equations (88) to (91) into (87) we obtain the L_q in closed form.

Further, we find the average system size L using Little's formula. Thus we have

$$L = L_q + \rho \quad (92)$$

where L_q has been found by equation (87) and ρ is obtained from equation (83).

Let W_q and W denote the mean waiting time in the queue and in the system respectively.

Then using Little's formula, we obtain,

$$W_q = \frac{L_q}{\lambda} \quad (93)$$

$$W = \frac{L}{\lambda} \quad (94)$$

where L_q and L have been found in equations (87) and (92)

6. Particular Cases

Case: 1

If there is no setup time, no feedback and compulsory vacation

(i.e., mean setup time $\frac{1}{\nu}=0$, $p=0, \theta=1$).

Then our model reduces to a single server $M^{[X]}/G/1$ queue with three stage heterogeneousservice, compulsory server vacation and service interruption. In this case we can find the idleprobability Q, utilisation factor ρ and average queue size L_q can be simlified to thefollowing expression.

$$Q = 1 - \lambda E(I)\left(\frac{1}{\beta \overline{B}(\alpha)} + \frac{1}{\alpha \overline{B}(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + E(V) \right) \quad (95)$$

$$\rho = \lambda E(I)\left(\frac{1}{\beta \overline{B}(\alpha)} + \frac{1}{\alpha \overline{B}(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + E(V) \right) \quad (96)$$

$$L_q = \left(\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \right) \quad (97)$$

where

$$N'(1) = \lambda Q E(I)\Big((\alpha+\beta)[1-\overline{B}(\alpha)] + \alpha\beta\overline{B}(\alpha)E(V) \Big) \quad (98)$$

$$N''(1) = Q[(\lambda E[I(I-1)](\alpha+\beta) + 2\lambda E(I)(-\lambda E(I)+\alpha))(1-\overline{B}(\alpha)) + 2(\lambda E(I))^2 (\overline{B'}_1(\alpha)\overline{B}_2(\alpha)\overline{B}_3(\alpha) + \overline{B}_1(\alpha)\overline{B'}_2(\alpha)\overline{B}_3(\alpha)$$
$$+ \overline{B}_1(\alpha)\overline{B}_2(\alpha)\overline{B'}_3(\alpha))[\alpha+\beta-\alpha\beta E(V)] + \overline{B}(\alpha)[\alpha\beta((\lambda E(I))^2 \theta E(V^2) + \lambda E[I(I-1)]E(V)) - 2(\lambda E(I))^2(\alpha+\beta)E(V)] \quad (99)$$

$$D'(1) = \Big(\alpha\beta\overline{B}(\alpha)[1 - E(V)\lambda E(I)] - \lambda E(I)(\alpha+\beta)[1-\overline{B}(\alpha)] \Big) \quad (100)$$

$$D''(1) = [(-\lambda E[I(I-1)](\alpha+\beta) + 2(\lambda E(I))^2)(1-\overline{B}(\alpha)) - 2(\lambda E(I))(\alpha+\beta)[1-\overline{B}(\alpha)E(V)\lambda E(I)]$$
$$+ (\overline{B'}_1(\alpha)\overline{B}_2(\alpha)\overline{B}_3(\alpha) + \overline{B}_1(\alpha)\overline{B'}_2(\alpha)\overline{B}_3(\alpha) + \overline{B}_1(\alpha)\overline{B}_2(\alpha)\overline{B'}_3(\alpha))$$
$$\times[-2(\lambda E(I))^2(\alpha+\beta) + 2(\lambda E(I))^2\alpha\beta E(V) - 2\alpha\beta\lambda E(I)] - \overline{B}(\alpha)[\alpha\beta[E(V^2)(\lambda E(I))^2 + E(V)\lambda E[I(I-1)]]] \quad (101)$$

Case: 2

If there is no setup time, no feedback and cumpulsory vacation

(i.e., mean setup time $\frac{1}{v} = 0$, $p = 0, \theta = 1$, $C(z) = z, E(I) = 1$, and $E(I(I-1)) = 0$).

Then our model reduces to a single server $M/G/1$ Queue with three stage heterogeneousservice with service interruption and compulsory server vacation. In this case we can findthe idle probability Q, utilisation factor ρ and average queue size L_q can be simlifiedto the following expression.

$$Q = 1 - \lambda\left(\frac{1}{\beta \overline{B}(\alpha)} + \frac{1}{\alpha \overline{B}(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + E(V) \right) \quad (102)$$

$$\rho = \lambda\left(\frac{1}{\beta \overline{B}(\alpha)} + \frac{1}{\alpha \overline{B}(\alpha)} - \frac{1}{\beta} - \frac{1}{\alpha} + E(V) \right) \quad (103)$$

$$L_q = \left(\frac{D'(1)N''(1) - N'(1)D''(1)}{2(D'(1))^2} \right) \quad (104)$$

where

$$N'(1) = \lambda Q\Big((\alpha+\beta)[1-\overline{B}(\alpha)] + \alpha\beta\overline{B}(\alpha)E(V) \Big) \quad (105)$$

$$N''(1) = Q[2\lambda(-\lambda+\alpha))(1-\overline{B}(\alpha)) + 2\lambda^2(\overline{B'}_1(\alpha)\overline{B}_2(\alpha)\overline{B}_3(\alpha)$$
$$+ \overline{B}_1(\alpha)\overline{B'}_2(\alpha)\overline{B}_3(\alpha) + \overline{B}_1(\alpha)\overline{B}_2(\alpha)\overline{B'}_3(\alpha))[\alpha+\beta-\alpha\beta E(V)] + \overline{B}(\alpha)[\alpha\beta(\lambda^2 E(V^2)) - 2\lambda^2(\alpha+\beta)E(V)] \quad (106)$$

$$D'(1) = \Big(\alpha\beta\overline{B}(\alpha)[1 - E(V)\lambda] - \lambda(\alpha+\beta)[1-\overline{B}(\alpha)] \Big) \quad (107)$$

$$D''(1) = [2\lambda^2(1-\overline{B}(\alpha)) - 2\lambda(\alpha+\beta)[1-\overline{B}(\alpha)E(V)\lambda] + (\overline{B'}_1(\alpha)\overline{B}_2(\alpha)\overline{B}_3(\alpha) + \overline{B}_1(\alpha)\overline{B'}_2(\alpha)\overline{B}_3(\alpha) + \overline{B}_1(\alpha)\overline{B}_2(\alpha)\overline{B'}_3(\alpha))$$

$$\times[-2\lambda^2(\alpha+\beta)+2\lambda^2\alpha\beta E(V)-2\alpha\beta\lambda]-\overline{B}(\alpha)[\alpha\beta E(V^2)\lambda^2]] \tag{108}$$

Case: 3

If there is no setup time, no feedback, no third stage service, compulsory vacation and single arrival (ie. $C(z)=z, E(I)=1, E(I(I-1))=0$ mean setup time $\frac{1}{v}=0$, $p=0,\theta=1,\overline{B}_3(\alpha)=1,\overline{B}'_3(\alpha)=0$).

Then our model reduces to a single server $M/G/1$ queue with two stage heterogeneous servicewithservice interruption and compulsory server vacation. In this case we can find the idleprobability Q, utilisation factor ρ and mean queue size L_q can be simlified to the followingexpression.

$$Q=1-\lambda\left(\frac{1}{\beta\overline{B}_1(\alpha)\overline{B}_2(\alpha)}+\frac{1}{\alpha\overline{B}_1(\alpha)\overline{B}_2(\alpha)}-\frac{1}{\beta}-\frac{1}{\alpha}+E(V)\right) \tag{109}$$

$$\rho=\lambda\left(\frac{1}{\beta\overline{B}_1(\alpha)\overline{B}_2(\alpha)}+\frac{1}{\alpha\overline{B}_1(\alpha)\overline{B}_2(\alpha)}-\frac{1}{\beta}-\frac{1}{\alpha}+E(V)\right) \tag{110}$$

$$L_q=\left(\frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2}\right) \tag{111}$$

where

$$N'(1)=\lambda Q\big((\alpha+\beta)[1-\overline{B}_1(\alpha)\overline{B}_2(\alpha)]+\alpha\beta\overline{B}_1(\alpha)\overline{B}_2(\alpha)E(V)\big) \tag{112}$$

$$N''(1)=Q[(1-\overline{B}_1(\alpha)\overline{B}_2(\alpha))(2\lambda(-\lambda+\alpha))+2\lambda^2(\overline{B}'_1(\alpha)\overline{B}_2(\alpha)$$

$$+\overline{B}_1(\alpha)\overline{B}'_2(\alpha))\times[\alpha+\beta-\alpha\beta E(V)]+\overline{B}_1(\alpha)\overline{B}_2(\alpha)[\alpha\beta(\lambda^2E(V^2)\ -2\lambda^2(\alpha+\beta)E(V)) \tag{113}$$

$$D'(1)=\big(\alpha\beta\overline{B}_1(\alpha)\overline{B}_2(\alpha)[1-E(V)\lambda]-\lambda(\alpha+\beta)[1-\overline{B}_1(\alpha)\overline{B}_2(\alpha)]\big) \tag{114}$$

$$D''(1)=[2\lambda^2(1-\overline{B}_1(\alpha)\overline{B}_2(\alpha))-2\lambda(\alpha+\beta)[1-\overline{B}_1(\alpha)\overline{B}_2(\alpha)E(V)\lambda]$$

$$+(\overline{B}'_1(\alpha)\overline{B}_2(\alpha)+\overline{B}_1(\alpha)\overline{B}'_2(\alpha))[-2\lambda^2(\alpha+\beta)+2\lambda^2\alpha\beta E(V)-2\alpha\beta\lambda]\ -\overline{B}_1(\alpha)\overline{B}_2(\alpha)[\alpha\beta[E(V^2)\lambda^2]] \tag{115}$$

The above equations are coincides with the resuts of V. Thangaraj and S. Vanitha (2010).

Case: 4

For this particular case we take $C(z)=1$ no feedback, no vacation, no setup time, no secondand third stage service and no breakdown and repair this means

$$p=0,\frac{1}{v}=0,\theta=0,\overline{B}_1(\alpha)=1,\overline{B}_2(\alpha)=1,\text{and }\ \alpha=\beta=0,\overline{B}_1(0)=1,\overline{B}'_1(0)=-E(B_1)\,.$$

$$W_q(z)=\left(\frac{(1-\lambda E(B_1)(\overline{B}_1(\lambda-\lambda z)-1))}{z-\overline{B}_1(\lambda-\lambda z)}\right) \tag{116}$$

We note that equation (89) is a known steady state result for the $M/G/1$ queue (see Kashyapand Chaudhry [7, equation (26), p. 58]).

Further, let $W(z)$ denote the probability generating function for the number of customer in thesystem, then we have

$$W(z)=Q+zW_q(z)\ =\big(1-\lambda E(B_1\big)+z\left(\frac{(1-\lambda E(B_1)(\overline{B}_1(\lambda-\lambda z)-1))}{z-\overline{B}_1(\lambda-\lambda z)}\right) \tag{117}$$

$$W(z)=\left(\frac{(1-\lambda E(B_1)(1-z)\overline{B}_1(\lambda-\lambda z)}{(z-\overline{B}_1(\lambda-\lambda z))}\right) \tag{118}$$

The result in equation (90) is the well-known Pollaczek-Khinchine formula (see Medhi [11 , equation (6.7) , p. 313] or Choi and Park [4 , p. 225]).

7. Numerical Example

The above queueing model is analysed numerically with the following assumptions.

1. Service time distribution is Erlang-k and the service rates are $\mu_1=10$, $\mu_2=9$, $\mu_3=8$.
2. Single arrival and the arrival rate ranging from $\lambda=0.1$ to 1.
3. Vacation time follows exponential distribution with $\theta=0.2$ and $\gamma=5$
4. Breakdown and repair time follows exponential distribution with parameters $\alpha=2$ and $\beta=8$
5. Feedback with probability $p=0.3(q=0.7)$
6. Setup time follows exponential distribution with parameter $\nu=6$

Results are presented for the values of $k=1,2,3,4.$ in the following tables.

Table 1. *Computed values of various queue characteristics when k=1.*

λ	Q	ρ	L_q	L	W_q	W
0.1	0.9048	0.0952	0.0394	0.1346	0.3938	1.3458
0.2	0.8127	0.1873	0.0933	0.2807	0.4667	1.4033
0.3	0.7235	0.2765	0.1665	0.4430	0.5550	1.4767
0.4	0.6371	0.3629	0.2656	0.6286	0.6640	1.5714
0.5	0.5533	0.4467	0.4011	0.8478	0.8023	1.6957
0.6	0.4721	0.5279	0.5899	1.1178	0.9831	1.8630
0.7	0.3933	0.6067	0.8610	1.4677	1.2300	2.0967
0.8	0.3168	0.6832	1.2696	1.9528	1.5871	2.4410
0.9	0.2425	0.7575	1.9343	2.6918	2.1492	2.9908
1.0	0.1704	0.8296	3.1645	3.9941	3.1645	3.9941

Table 2. *Computed values of various queue characteristics when k=2.*

λ	Q	ρ	L_q	L	W_q	W
0.1	0.8998	0.1002	0.0410	0.1412	0.4097	1.4120
0.2	0.8028	0.1972	0.0981	0.2953	0.4903	1.4764
0.3	0.7089	0.2911	0.1768	0.4679	0.5893	1.5598
0.4	0.6179	0.3821	0.2856	0.6677	0.7140	1.6693
0.5	0.5297	0.4703	0.4378	0.9081	0.8757	1.8163
0.6	0.4442	0.5558	0.6563	1.2121	1.0938	2.0201
0.7	0.3612	0.6388	0.9828	1.6216	1.4040	2.3166
0.8	0.2807	0.7193	1.5044	2.2237	1.8806	2.7797
0.9	0.2025	0.7975	2.4353	3.2328	2.7059	3.5920
1.0	0.1266	0.8734	4.4841	5.3575	4.4841	5.3575

Table 3. *Computed values of various queue characteristics when k=3.*

λ	Q	ρ	L_q	L	W_q	W
0.1	0.8979	0.1021	0.0416	0.1437	0.4156	1.4367
.2	0.7991	0.2009	0.0998	0.3007	0.4992	1.5037
0.3	0.7034	0.2966	0.1807	0.4773	0.6024	1.5911
0.4	0.6107	0.3893	0.2933	0.6826	0.7333	1.7065
0.5	0.5209	0.4791	0.4523	0.9314	0.9046	1.8628
0.6	0.4338	0.5662	0.6831	1.2493	1.1385	2.0822
0.7	0.3493	0.6507	1.0338	1.6845	1.4769	2.4065
0.8	0.2673	0.7327	1.6079	2.3407	2.0099	2.9258
0.9	0.1876	0.8124	2.6762	3.4885	2.9735	3.8761
1.0	0.1102	0.8898	5.2427	6.1325	5.2427	6.1325

Table 4. *Computed values of various queue characteristics when k=4.*

λ	Q	ρ	L_q	L	W_q	W
0.1	0.8969	0.1031	0.0419	0.1450	0.4188	1.4496
0.2	0.7972	0.2028	0.1008	0.3036	0.5039	1.5181
0.3	0.7006	0.2994	0.1828	0.4822	0.6094	1.6075
0.4	0.6070	0.3930	0.2974	0.6904	0.7436	1.7261
0.5	0.5163	0.4837	0.4601	0.9438	0.9201	1.8875
0.6	0.4284	0.5716	0.6976	1.2693	1.1627	2.1154
0.7	0.3431	0.6569	1.0618	1.7188	1.5169	2.4554
0.8	0.2602	0.7398	1.6662	2.4060	2.0827	3.0074
0.9	0.1798	0.8202	2.8178	3.6379	3.1309	4.0422
1.0	0.1017	0.8983	5.7357	6.6339	5.7357	6.6339

8. Conclusion

In this paper we studied Three Stage Heterogeneous Service with Feedback Queue and Service Interruption, Setup Time with Bernoulli Vacation. The single server provides three stages of hetereogeneous service to the each customers. We derived the probability generating functions of the number of customers in the queue are found by using the supplementary variable technique, average queue size, the average waiting time for the customers and numerical results are also obtained. In the numerical results, it clearly shows as long as increasing the arrival rate, The server's idle time decreases while the utilization farcor, average queue size, system size and average waiting time in the queue and system of our queueing model are all increases.

References

[1] Ayyappan. G and Shyamala. S, $M^{[X]}/G_1, G_2/1$ with setup time, Bernoulli vacation, Breakdown, and DelayedRepair, *International* Journal ofStochastic Analysis, (2014).

[2] Chae, K. C. Lee, H. W. and Ahn, C. W. An arrival time approach to M/G/1 type queues with generalisedvacations, *Queueing Systems*, (2001) vol. 38, pp. 91-100.

[3] Choudhury. G, An $M^{[X]}/G/1$ queueing system with a setup periodand a vacation period, *QueueingSystems*, (2000) vol. 36, pp. 23-38.

[4] Choudhury. G, A note on the $M^{[X]}/G/1$ queue with a random set-up time under a restricted admissibility policywith a Bernoullischedule vacation, *Statistical Methodology*, (2008) vol. 5, pp. 21-29.

[5] Doshi, B. T., Queueing systems with vacation-a survey, *Queueing Systems*, (1986) vol. 1, No. 1, pp. 29-66.

[6] Gross. C and Harris. C. M, The fundamentals of queuing theory, (1985) second ed., John Wiley and Sons, New York.

[7] Kashyap. B. R. K, Chaudhry. M. L, An introduction to queuing theory, *A and A publications*, Kingston, Ont., Canada (1988).

[8] Madan. K. C, Choudhury. G, An $M^{[X]}/G/1$ queue with a Bernoulli schedule vacation under restrictedadmissibility policy, *Sankhya*, (2004) vol. 66, no: 1, pp. 175-193.

[9] Madan. K. C and Rawwash. M. A, On the $M^{[X]}/G/1$ queue withfeedback and optional server vacationsbased on a single vacation policy, *Applied Mathematics and Computation*, (2005) vol. 160, pp. 909-919.

[10] Maragatha Sundari. S and Srinivasan. S, Time dependent solution of a Non-Morkovian queue with triple stagesof services having compulsory vacation and service intereptions, *International Journal of Computer Applications*, (2012) Vol. 41, No. 7, pp. (0975-8887).

[11] Maragathasundari, S., Karthikeyan, K. Batch Arrival Of Two Stages With Standby ServerDuring General Vacation Time & General Repair Time *International Journal of Mathematical Archive* (2015a), Vol. 6 (4), pp. 43-48.

[12] Maragathasundari, S. and Karthikeyan, K. A Study on M/G/1 Queueing System with ExtendedVacation, Random Breakdowns and General Repair *International Journal of Mathematics And its Applications* (2015b), Vol. 3, pp. 43-49.

[13] Medhi. J, Stochastic Processes, *Wiley*, Eastern (1982).

[14] Rehab F. Khalaf, Kailash C. Madan and Cormac A. Lukas An$M^{[X]}/G/1$ queue with Bernoulli schedulegeneral vacation times, general extended vacations, random breakdowns, general delay times for repairs to start andgeneral repair times, *Journal of MathematicsResearch*, (2011) Vol. 3, no. 4.

[15] Takagi, H. Time - dependent analysis of an M/G/1 vacation models with exhaustive service, *QueueingSystems*, (1990) vol. 6, No. 1, pp. 369-390.

[16] Thangaraj. V and Vanitha. S, A single server M/G/1 feedback queue with two types of service having generaldistribution, *International Mathematical Forum*, (2010a) vol. 5, no. 1, pp. 15-33.

[17] Thangaraj. V and Vanitha. S An M/G/1 queue with two-stage heterogeneous service compulsory servervacation and random breakdowns, *Int. J. Contemp. Math. Sciences*, (2010b) Vol. 5, no. 7, pp. 307-322.