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Thermomagnetic Convection in a Ferrofluid Layer: Effects of Non-Uniform Basic Temperature Profiles and MFD Viscosity

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Abstract: The combined effect of magnetic field dependent (MFD) viscosity and non-uniform basic temperature profiles on the onset of thermomagnetic convection in a horizontal ferrofluid layer is studied analytically using linear stability theory, The lower and upper boundaries of the ferrofluid layer are assumed to be rigid by prescribing uniform heat flux condition at the lower boundary and a general thermal condition at the upper boundary. The Galerkin technique is used to find the eigenvalues as this technique is found to be more convenient to tackle different forms of basic temperature profiles. The results indicate that the basic cubic temperature profiles have a profound influence on the stability characteristics of the system and can be effectively used to either suppress or augment the onset of thermomagnetic convection. Results show that the conductive effect of non-steady conditions within the fluid layer does play a stabilizing state. It is observed that the effect of magnetic number, nonlinearity of the fluid magnetization is to hasten, while an increase in the magnetic field dependent viscosity parameter and Biot number is to delay the onset of ferroconvection.

Keywords: Non-Uniform-Basic-Temperature Profiles, Ferroconvection, Magnetic-Fiield-Density Viscosity, Galerkin Technique, Darcy-Rayleigh Number, General Thermal Boundaries

1. Introduction

Ferrofluids or magnetic fluids are commercially manufactured colloidal liquids usually formed by suspending mono domain nanoparticles (their diameter is typically $3\sim10$ nm) of magnetite in non-conducting liquids like heptane, kerosene, water etc and they are also called magnetic nanofluids. These fluids get magnetized in the presence of an external magnetic field and due to their both liquid and magnetic properties they have emerged as reliable materials capable of solving complex engineering problems. An authoritative introduction to this fascinating subject and along with the important applications of these fluids in many practical problems is well documented in the literature [1-5]. It is also recognized that these fluids have promising potential for heat transfer applications in electronics, micro and nano-electro-mechanical systems (MEMS and NEMS), and air-conditioning and ventilation systems for details see Ganguly et al. [6].

The magnetization of ferrofluids depends on the magnetic field, temperature, and density. Hence, any variations of these quantities induce change of body force distribution in the fluid and eventually give rise to convection in ferrofluids in the presence of a gradient of magnetic field. There have been numerous studies on thermal convection in a ferrofluid layer called ferroconvection analogous to Rayleigh-Benard convection in ordinary viscous fluids. Finlayson [7] was the first to study thermal convective instability in a layer of ferrofluid heated from below in the presence of a uniform vertical magnetic field. Since then several studies have been undertaken in this direction to understand heat transfer in ferrofluids [8-18]. Sunil and Mahajan [19] have performed nonlinear stability analysis for a magnetized ferrofluid layer heated from below for the stress-free boundaries case, while Nanjundappa and Shivakumara [20] have investigated the effects of different kinds of velocity and temperature boundary conditions on the onset of ferroconvection in an initially quiescent ferrofluid layer. Recently, thermal convection of ferrofluids in the presence of a uniform vertical magnetic field with the boundary temperatures modulated sinusoidally about some reference values has been discussed by Jitender Singh and Renu Bajaj [12], while Belyaev and Smorodin [13] have studied the effect of an alternating uniform magnetic field on the onset of convection in a horizontal layer of a ferrofluid within the framework of a quasi-stationary approach.

Moreover, many physical properties of ferrofluids can be tuned by varying the magnetic field. One of the well known phenomena generated by the influence of magnetic fields on ferrofluids is the change of their viscous behavior. Rosensweig et al. [21] were the first to discover such changes in concentrated magnetite-ferrofluids and by dimensional analysis they predicted that the viscosity to be a function of the ratio of hydrodynamic stress to magnetic stress. Realizing the importance of magnetic field dependent (MFD) viscosity on ferrofluid flows, several studies have been undertaken in the past. The effect of a homogeneous magnetic field on the viscosity of a fluid with solid particles possessing intrinsic magnetic moments has been investigated by Shliomis [22]. The effect of MFD viscosity on the onset of ferroconvection in a rotating ferrofluid layer is discussed by Vaidyanathan et al. [23]. Recently, Nanjundappa et al. [24] have investigated the effect of MFD viscosity on the onset of convection in a ferromagnetic fluid layer in the presence of a uniform vertical magnetic field by considering the bounding surfaces are either rigid-ferromagnetic or stress- free with constant heat flux conditions.

It is a well established fact that convection can also be induced by surface-tension forces provided it is a function of temperature. In view of the fact that heat transfer is greatly enhanced due to convection, the magnetic convection problems offer new possibilities for new applications in cooling with motors, loud speakers, transmission lines, and other equipment where magnetic field is already present. If the ferrofluid layer has an upper surface open to atmosphere then the instability is due to the combined effects of the buoyancy as well as temperature-dependent surface tension forces, known as Bénard-Marangoni ferroconvection. A limited number of studies have addressed the effect of surface tension forces on ferroconvection in a horizontal ferrofluid layer. Linear and non-linear stability of combined buoyancy-surface tension effects in a ferrofluid layer heated from below is considered by Qin and Kaloni [14]. The coupling between Marangoni and Rosensweig instabilities by considering two semi-infinite incompressible and immiscible viscous fluids of infinite lateral extent in which one of them is ferromagnetic and the other is a usual Newtonian liquid is studied by Weiplepp and Brand [15]. Shivakumara et al. [16] have investigated the effect of different forms of basic temperature gradients on the onset of ferroconvection driven by combined surface tension and buoyancy forces with an idea of understanding control of ferroconvection. The Rayleigh-Bénard-Marangoni instability in a ferrofluid layer in the presence of weak vertical magnetic field normal to the boundaries has been discussed by Hennenberg et al. [17]. The onset of Marangoni ferroconvection with different initial temperature gradients is analyzed by Shivakumara and Nanjundappa [18].

Thermal convection in ferromagnetic fluids is gaining much importance due to its astounding physical properties. One such property is viscosity of the ferromagnetic fluid. The viscosity of the ferrofluid is predicted by dimensional analysis to be a function of the ratio of hydrodynamic stress to magnetic stress (Rosenswieg et al. [19]). The effect of a homogeneous magnetic field on the viscosity of a fluid with solid particles possessing intrinsic magnetic moments has been investigated by Shliomis [20]. The effect of magnetic field dependent (MFD) viscosity on the onset of ferroconvection in a rotating ferrofluid layer is discussed by Vaidaynathan et al. [21], with or without dust particles by Sunil et al. [22] and the non-linear stability analysis has also been performed by Sunil et al. [23]. Recently, Nanjundappa et al. [24] have investigated the effect of MFD viscosity on the onset of convection in a ferromagnetic fluid layer in the presence of a vertical magnetic field by considering the bounding surfaces are either rigid-ferromagnetic or stress-free with constant heat flux conditions. In this sense more investigations have been taken place see [25]-[28].

The intent of the present paper is to study coupled Bénard-Marangoni ferroconvection in a ferrofluid layer in the presence of a uniform vertical magnetic field with magnetic field dependent viscosity. The lower boundary is rigid with fixed temperature, while the upper non-deformable free boundary is subjected to temperature dependent surface tension forces and a general thermal boundary condition on the perturbation temperature is imposed. The study helps in understanding control of ferroconvection by magnetic field dependent viscosity, which is useful in many heat transfer related problems particularly in materials science processing. The resulting eigenvalue problem is solved numerically by employing the Rayleigh-Ritz method with Chebyshev polynomials of the second kind as trial functions.

The paper is organized as under. Section 2 is devoted to the formulation of the problem. The method of solution is discussed in Section 3. In Section 4, the numerical results are discussed and some important conclusions follow in Section 5.

2. Mathematical Formulation

The present paper considereda Boussinesq ferrofluid layer of thickness d with no lateral boundaries and auniform magnetic field H_0 acting normal to the boundaries. The lower and the upper boundaries are maintained at constant but different temperatures T_0 and T_1 ($< T_0$) respectively. A Cartesian co-ordinate system (*x, y, z*) is used with the origin at the lower boundary and the z-axis vertically upward. Gravity acts in the negative z-direction, $\vec{g} = -g\,\hat{k}$, where $\hat{k}$ is the unit vector in the z-direction. The layer is bounded below by a rigid surface while the free surface which is subjected to temperature dependent surface tension forces is assumed to be flat and non deformable. The fluid density ρ is assumed to vary linearly with temperature in the form

$$\rho=\rho_0\left[1-\alpha_t\,(T-T_0)\right] \tag{1}$$

where α_t is the thermal expansion coefficient and ρ_0 is the density at $T=T_0$.

In the study of ferroconvection, we have to solve the Maxwell equations simultaneously with the balance of mass, linear momentum and energy. Since the fluid is assumed to be electrically not conducting, the Maxwell equations reduce to

$$\nabla\cdot\vec{B}=0 \tag{2}$$

$$\nabla\times\vec{H}=0 \tag{3}$$

In view of Eq. (3), the magnetic field expressed by a scalar potential

$$\vec{\mathrm{H}}=\nabla\varphi \tag{4}$$

Further $\vec{B}$, $\vec{M}$ and $\vec{H}$ are related by

$$\vec{B}=\mu_0\left(\vec{M}+\vec{H}\right) \tag{5}$$

By assuming that the magnetization is aligned with the magnetic field, but allow dependence on the magnitude of magnetic field as well as on the temperature, thus

$$\vec{M}=[M_0+\chi(H-H_0)-K(T-T_0)]\left(\vec{H}/H\right) \tag{6}$$

The momentum equation is

$$\rho_0\left[\frac{\partial\vec{q}}{\partial t}+(\vec{q}\cdot\nabla)\,\vec{q}\right]=-\nabla\mathrm{p}+\rho\,\vec{g}+\mu_0\,(\vec{M}\cdot\nabla)\,\vec{H}+2\nabla\cdot[\eta\,\underset{\sim}{D}] \tag{7}$$

The fluid is assumed to be incompressible having variable viscosity. Experimentally, it has been demonstrated that the magnetic viscosity has got exponential variation, with respect to magnetic field (Rosenswieg [19]). As a first approximation, for small field variation, linear variation of magnetic viscosity has been used in the form $\eta=\eta_0(1+\vec{\delta}\cdot\vec{B})$, where $\vec{\delta}$ is the variation coefficient of magnetic field dependent viscosity and is considered to be isotropic (Vaidyanathan et al. [21]), η_0 is taken as viscosity of the fluid when the applied magnetic field is absent.

Neglecting viscous dissipation, the energy equation is

$$\left[\rho_0 C_{V,H}-\mu_0\vec{H}\cdot\left(\frac{\partial\vec{M}}{\partial T}\right)_{V,H}\right]\frac{DT}{Dt}+\mu_0 T\left(\frac{\partial\vec{M}}{\partial T}\right)_{V,H}\cdot\frac{D\vec{H}}{Dt}=k_t\nabla^2 T \tag{8}$$

The continuity equation is

$$\nabla\cdot\vec{q}=0 \tag{9}$$

The basic state is quiescent and is given by

$$\vec{q}=0\,,\; p=p_b(z)\,,-\frac{\partial T_b}{\partial z}=f(z),\; \vec{H}_b=\left[H_0-\frac{K\beta z}{1+\chi}\right]\hat{k}\,,\vec{M}_b=\left[M_0+\frac{K\beta z}{1+\chi}\right]\hat{k} \tag{10}$$

where, $\hat{k}$ is the unit vector in the z-direction and $f(z)$ is the basic temperature gradient, such that $\int_0^d f(z)\,dz=\frac{\Delta T}{d}$.

The stability of the basic state can be analyzed by introducing the following perturbations:

$$\vec{q}=\vec{q}',\; p=p_b(z)+p',\; \eta=\eta_b(z)+\eta',\, T=T_b(z)+T',\; \vec{H}=\vec{H}_b(z)+\vec{H}',\vec{M}=\vec{M}_b(z)+\vec{M}' \tag{11}$$

where, $\vec{q}'$, p', η', T', $\vec{H}'$ and $\vec{M}'$ are perturbed variables and are assumed to be small.

Substituting Eq. (11) into Eq. (2) and using Eqs. (5) and (6), the following obtained (after dropping the primes)

$$H_x+M_x=\left(1+M_0/H_0\right)H_x\,,H_y+M_y=\left(1+M_0/H_0\right)H_y\,,H_z+M_z=\left(1+\chi\right)H_z-K\,T\,. \tag{12}$$

where, (H_x, H_y, H_z) and (M_x, M_y, M_z) are (x, y, z) components of perturbed magnetic field and magnetization respectively. In obtaining the above equations, it is assumed that $K\beta\,d<<\left(1+\chi\right)H_0$.

Substituting Eq. (11) into Eq. (7) and linearizing, we obtain in components (after neglecting the primes)

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \eta_0 \left[1 + \mu_0 \delta (M_0 + H_0)\right] \nabla^2 u + \mu_0 (M_0 + H_0) \frac{\partial H_x}{\partial z} \tag{13}$$

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \eta_0 \left[1 + \mu_0 \delta (M_0 + H_0)\right] \nabla^2 v + \mu_0 (M_0 + H_0) \frac{\partial H_y}{\partial z} \tag{14}$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \rho_0 \alpha_t g T + \eta_0 \left[1 + \mu_0 \delta (M_0 + H_0)\right] \nabla^2 w + \mu_0 (M_0 + H_0) \frac{\partial H_z}{\partial z} - \mu_0 K f(z) H_z + \frac{\mu_0 K^2 f(z) T}{1+\chi} \tag{15}$$

Differentiating Eqs. (13) and (14) partially with respect to x and y respectively and adding, obtained following

$$\nabla_1^2 p = -\rho_0 \alpha_t g \frac{\partial T}{\partial z} + \mu_0 (M_0 + H_0) \frac{\partial}{\partial z} (\nabla \cdot \vec{H}) - \mu_0 K f(z) \frac{\partial H_z}{\partial z} + \frac{\mu_0 K^2 f(z)}{1+\chi} \frac{\partial T}{\partial z} \tag{16}$$

where $\nabla_1^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the horizontal Laplacian operator. Eliminating the pressure termfrom Eq. (15), using Eq. (16), finally

$$\left[\rho_0 \frac{\partial}{\partial t} - \eta_0 \left\{1 + \delta \mu_0 (M_0 + H_0)\right\} \nabla^2 \right] \nabla^2 w = -\rho_0 \alpha_t g \nabla_1^2 T + \mu_0 K f(z) \frac{\partial}{\partial z} (\nabla_1^2 \phi) + \frac{\mu_0 K^2 f(z)}{1+\chi} (\nabla_1^2 T) \tag{17}$$

As before, substituting Eq. (11) into Eq. (8) and linearizing (after neglecting primes)

$$\rho_0 C_0 \frac{\partial T}{\partial t} - \mu_0 K T_0 \frac{\partial}{\partial t} \left(\frac{\partial \varphi}{\partial z} \right) = \left(\rho_0 C_0 - \frac{\mu_0 K^2 T_0}{1+\chi} \right) w f(z) + k_t \nabla^2 T \tag{18}$$

where $\rho_0 C_0 = \rho_0 C_{V,H} + \mu_0 K H_0$.

Finally, Eqs. (2) and (3), after using Eqs. (12) and (13), yield (after neglecting primes)

$$\left(1 + \frac{M_0}{H_0}\right) \nabla_1^2 \phi + (1+\chi) \frac{\partial^2 \phi}{\partial z^2} - K \frac{\partial T}{\partial z} = 0. \tag{19}$$

Since by assuming that the principle of exchange of stability is valid, the normal mode solution in the form

$$\{\mathrm{w, T, \varphi}\} = \{\mathrm{W}, \Theta, \Phi\}(z)\, Exp[\mathrm{i}(\mathrm{l x + m y}) + \omega t] \tag{20}$$

where, l and m are wavenumbers in the x and y directions, $\mathrm{W(z)}$, $\Theta(z)$, $\Phi(z)$ are the amplitude of z-component of perturbation velocity, perturbation temperature, perturbation magnetization and ω is the growth rate. Substituting Eq. (20) in Eqs. (17)- (19) and non-dimensionalizing the quantities in the form

$$(x^*, y^*, z^*) = \left(\frac{x}{d}, \frac{y}{d}, \frac{z}{d}\right), \quad W^* = \frac{d}{\nu} W, \ \mathrm{t}^* = \frac{\nu}{d^2} t, \quad \Theta^* = \frac{\kappa}{\beta\, \nu\, d} \Theta, \ \Phi^* = \frac{(1+\chi)\ \kappa}{K\, \beta\, \nu\, d^2} \Phi,$$

$$\delta^* = \mu_0 (M_0 + H_0) \delta, \ f(z)^* = \frac{1}{\beta} f(z) \left(\beta = -\frac{\Delta T}{d}\right) \tag{21}$$

obtained (after ignoring the asterisks)

$$\left[(1+\Lambda)\left\{(D^2 - a^2) - \omega\right\}\right](D^2 - a^2) W = R a^2 \Theta - N a^2 f(z) [\mathrm{D}\Phi - \Theta] \tag{22}$$

$$\left(D^2 - a^2 - \Pr \omega\right) \Theta - \Pr M_2 \omega D \Phi = -(1 - M_2) f(z) W \tag{23}$$

$$\left(\mathrm{D}^2 - \mathrm{a}^2 \mathrm{M}_3\right) \Phi - D\Theta = 0. \tag{24}$$

$f(z)$ is the non-dimensional temperature gradient such that $\int_0^1 f(z)\, dz = 1$.

The boundary conditions for the perturbed non-dimensional variables take the form

$$W = DW = D\Theta = \Phi = 0 \text{ at } z = 0 \quad (25)$$

$$W = DW = D\Theta + Bi\,\Theta = \Phi = 0 \text{ at } z = 1 \quad (26)$$

The case $Bi = 0$ and $Bi \to \infty$ respectively correspond to constant heat flux and isothermal conditions at the upper boundary.

TheBoussinesq approximation is assumed (except for the surface tension) to be varying with the temperature. The steady state temperature profile reveals the form of a cubic equation within the fluid layer due to the existence of the non-steady conditions (Pearson (1958), Perea-Garcıa (1991) and Dupont (1992)), given by

$$T_b = T_{0s} - a_1(z-d) - a_2(z-d)^2 - a_3(z-d)^3,$$

where T_{0s} is the temperature at the upper free surface of the fluid layer and $a_{i,}\ i = 1,2,3$ is the polynomial coefficient of the transient temperature profile. In non-dimensional form, the $f(z)$ in this case is given by

$$f(z) = a_1^* + 2a_2^*\ (z-1) + 3a_3^*\ (z-1)^2.$$

The special case $a_1^* = 1,\ a_2^* = 0$ and $a_3^* = 0$ recovers the classical linear basic state temperature distribution. The different temperature gradients studied in this paper are listed in Table 1.

Table 1. *Reference steady-state temperature gradients.*

Model	Temperature gradient	$f(z)$	a_1^*	a_2^*	a_3^*
1	Linear	1	1	0	0
2	Inverted Parabolic	$2(1-z)$	0	-1	0
3	Parabolic	$2z$	2	1	0
4	Cubic 1	$2(z-1)^2$	0	0	1
5	Cubic 2	$0.66+1.02(z-1)^2$	0.66	0	0.34

3. Method of Solution

Equations (22)-(24) together with boundary conditions given by Eqs. (25) – (26) constitute an eigenvalue problem with thermal Rayleigh number R_t being an eigenvlaue. Accordingly, W, Θ and Φ are written as

$$W(z) = \sum_{i=1}^{n} A_i\, W_i(z)\,,\ \Theta(z) = \sum_{i=1}^{n} B_i\ \Theta_i(z) \text{ and}$$

$$\Phi(z) = \sum_{i=1}^{n} C_i\ \Phi_i(z) \quad (27)$$

where A_i, B_i and C_i are unknown constants to be determined. The basis functions $W_i(z)$, $\Theta_i(z)$ and $\Phi_i(z)$ are generally chosen such that they satisfy the corresponding boundary conditions. Substituting Eq. (27) into Eqs. (22)–(24), multiplying the resulting momentum Eq. (22) by $W_j(z)$, energy Eq. (23) by $\Theta_j(z)$ and the magnetic potential Eq. (24) by $\Phi_j(z)$; performing the integration by parts with respect tozbetween $z = 0$ and $z = 1$ and using the boundary conditions (25) – (26), the following system of linear homogeneous algebraic equations are obtained:

$$C_{ji}A_i + D_{ji}B_i + E_{ji}C_i = 0 \quad (28)$$

$$F_{ji}A_i + G_{ji}B_i = 0 \quad (29)$$

$$H_{ji}B_i + I_{ji}C_i = 0\,. \quad (30)$$

The coefficients $C_{ji} - I_{ji}$ involve the inner products of the basis functions and are given by

$$C_{ji} = (1+\Lambda) < D^2W_jD^2W_i + 2a^2DW_jDW_i + a^4W_jW_i >$$

$$D_{ji} = -a^2 R_t\,[< W_j\,\Theta_i > + M_1 < f(z)W_j\,\Theta_i >]$$

$$E_{ji} = a^2 R_m < f(z)W_jD\Phi_i >$$

$$F_{ji} = -< f(z)\Theta_jW_i >$$

$$G_{ji} = < D\Theta_jD\Theta_i > + a^2 < \Theta_j\Theta_i > + Bi\ \Theta_j(1)\ \Theta_i(1)$$

$$H_{ji} = -< D\Phi_j\Theta_i >$$

$$I_{ji} = < D\Phi_jD\Phi_i > + a^2M_3 < \Phi_j\Phi_i > \quad (31)$$

where the inner product is defined as $< \cdots > = \int_0^1 (\cdots)\,dz.$

The above set of homogeneous algebraic equations can have a non-trivial solution if and only if

$$\begin{vmatrix} C_{ji} & D_{ji} & E_{ji} \\ F_{ji} & G_{ji} & 0 \\ 0 & H_{ji} & I_{ji} \end{vmatrix} = 0. \quad (32)$$

The eigenvalue has to be extracted from the above characteristic equation. For this, selected the following trial functions

$$W_i = (z^4 - 2z^3 + z^2)T_{i\text{-}1}^*,\ \Theta_i = z^2(1 - 2z/3)T_{i\text{-}1}^* \text{ and}$$

$$\Phi_i = (z^2 - z)(z-2)T_{i\text{-}1}^*\,. \quad (33)$$

where, T_i^* sare the Chebyshev polynomials of the second kind, such that W_i, Θ_i and Φ_i satisfy the corresponding boundary conditions except, $(1+\Lambda)D^2W + Ma\,a^2\,\Theta = D\Theta + Bi\,\Theta\ = 0$ at $z = 1$ but the residuals from the equations are included as residuals from the differential equations. Equation (32) leads to a relation involving the physical parameters R_t, Λ, M_1, M_3, Bi, a_1^*, a_2^*, a_2^* and a in the form

$$f(R_t, \Lambda, Bi, M_1, M_3, a_1^*, a_2^*, a_2^* \; a) = 0. \quad (34)$$

The critical value of R_t (i.e., R_{tc}) is determined numerically with respect to a for different values of Λ, M_1, M_3, Bi, a_1^*, a_2^* and a_2^*.

4. Results and Discussion

The critical stability parameters are obtained numerically using the Galerkin method with general types of boundary conditions. Besides, analytical solutions are also obtained using a regular perturbation technique when the boundaries are kept at constant heat flux conditions and the results are compared with those obtained numerically. A discussion on the results is made in the following sub-sections.

4.1. Numerical Solution for General Thermal Conditions

The linear stability theory is used to investigate the effect of MFD viscosity and non-uniform basic temperature profile on the onset of ferroconvection in a horizontal ferrofluid in the presence of a uniform vertical magnetic field. The lower and upper boundaries of the ferrofluid layer are assumed to be rigid by prescribing uniform heat flux condition at the lower boundary and a general thermal condition at the upper boundary. The resulting eigenvalue problem is solved numerically using the Galerkin method. First checked the numerical procedure used by comparing the critical Rayleigh number and the corresponding wave number with those of Sparrow et al. [34] for an ordinary viscous fluid. It is observed that six terms (i.e., $N = 6$) in the series expansion of Eq. (34) are required to obtain convergent results.

Table 2. *Comparison of R_c and a_c for different values of Bi with $M_1 = 0 = \Lambda$ and $M_3 = 1$.*

Bi	**Sparrow et. al [36] Rigid-Rigid R_c a_c**	**Present analysis Rigid-Rigid R_c a_c**
0	720. 000 0.00	720.000 0.000
0.01	747.765 0.71	747.765 0.7126
0.03	768.153 0.93	768.155 0.9283
0.1	807.676 1.23	807.676 1.2281
0.3	869.231 1.57	869.208 1.5571
1	974.173 1.94	974.172 1.9427
3	1093.744 2.24	1093.74 1.2419
10	1204.571 2.44	1204.57 2.4367
30	1259.884 2.51	1259.91 2.5110
100	1284.263 2.53	1284.28 2.5394
∞	1295.781 2.55	1295.78 2.5490

Table 2 shows the comparison of our results with those of Sparrow et al. [36] for different values of Bi under the limiting case of $M_1 = M_3 = 0$, $\Lambda = 0$, $a_1^* = 1$, $a_2^* = 0$ and $a_3^* = 0$ (i.e., ordinary viscous fluid). From the table it is evident that there is an excellent agreement between the results of the present study and the previously published ones. This verifies the applicability and accuracy of the method used in solving the problem.

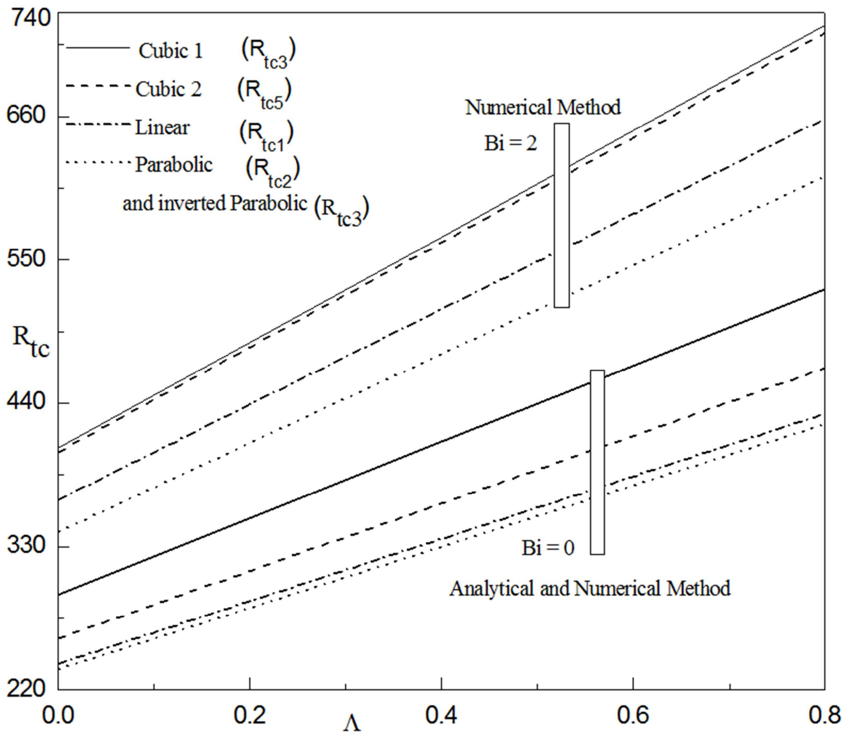

Figure 1. *The variation of R_{tc} as a function of Λ for different non-linear temperature profiles with two values of Bi when $M_1 = 2$ and $M_3 = 1$.*

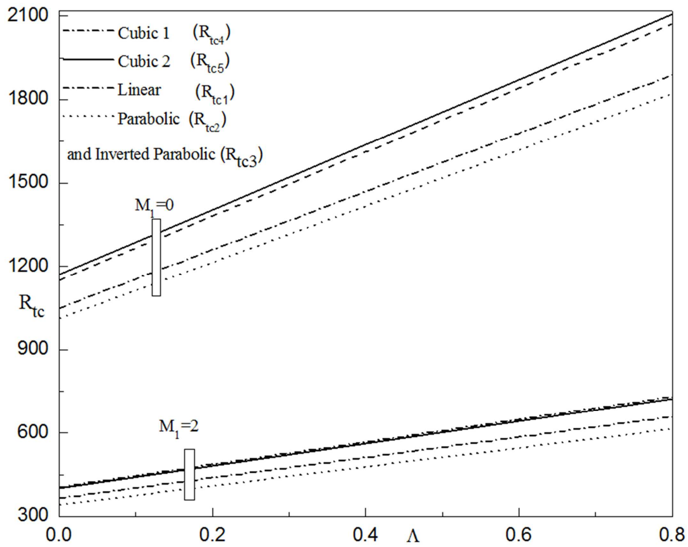

Figure 2. *The variation of* R_{tc} *as a function of* Λ *for different non-linear temperature profiles with two values of* M_1 *when* $Bi = 2$ *and* $M_3 = 1$.

The critical Rayleigh numbers R_{tc} and the corresponding wave numbers a_c obtained for different values of Λ, Bi, M_1 and M_3 are presented graphically in Figures 1-3. Figure 1 shows the variation of critical Rayleigh number R_{tc} as a function of MFD viscosity parameter Λ for different types of non-uniform basic temperature profile with $M_1 = 2$ and $M_3 = 1$. From the figure it is evident that the variation of Bi from 0 to 2 significantly increase the critical Rayleigh numbers in both the cases of velocity boundary conditions considered; the least being for $Bi = 0$ and the highest values correspond to those for $Bi = 2$. Thus the system is found to be more unstable for the upper heat insulating boundary as compared to isothermal condition at the upper boundary. This behavior is not surprising as the nature of the upper boundary changes drastically from an insulated surface to a conductive boundary with an increase in the value of Bi. It is evident that with an increase in the value of Bi the temperature perturbations will not grow so easily and therefore higher values of R_{tc} are needed for the onset of ferroconvection. Further, the critical Rayleigh number R_{tc} increase with an increase in the MFD viscosity parameter Λ and thus it has a stabilizing effect on the system. That is, the effect of increasing Λ is to delay the onset of ferroconvection. Besides, for a fixed value of Bi, the critical Rayleigh numbers (i.e., R_{tc1}) for cubic 1 temperature profile (i.e., Model 3) with $a_1^* = 0 = a_2^*$ and $a_3^* = 1$ is shown to be the most stabilizing of all the considered types of temperature profiles, that is, $R_{tc2}(= R_{tc3}) < R_{tc1} < R_{tc4} < R_{tc3}$.

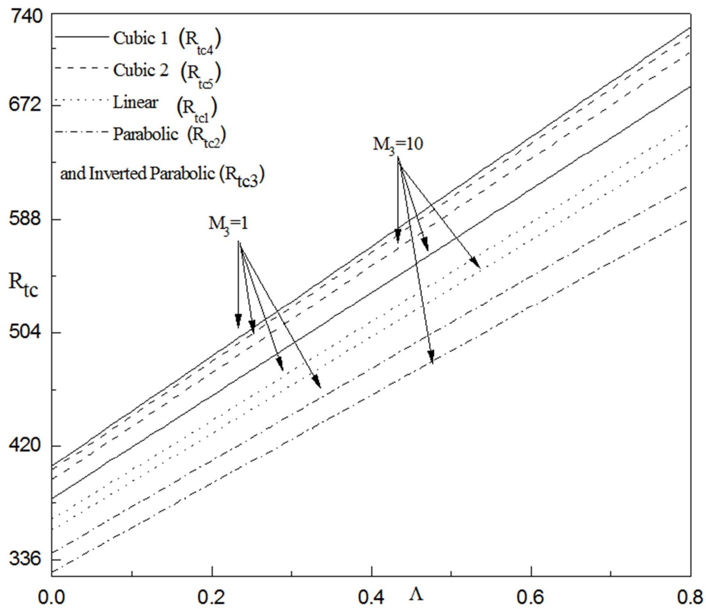

Figure 3. *The variation of* R_{tc} *as a function of* Λ *for different non-linear temperature profiles with two values of* M_3 *when* $Bi = 2$ *and* $M_1 = 2$.

The measure of nonlinearity of fluid magnetization, denoted through the parameter M_3, on the onset of ferroconvection is depicted in Figure 2. The curves of R_{tc} versus Λ shown in Figure 3 for different values of M_3 with non-uniform temperature profiles when $Bi = 2$ and $M_1 = 2$, demonstrate that increasing M_3 has a destabilizing effect on the system. Nevertheless, the destabilization due to increase in the nonlinearity of the fluid magnetization is only marginal. This may be attributed to the fact that a higher value of M_3 would arise either due to a larger pyromagnetic coefficient or larger temperature gradient. Both these factors are conducive for generating a larger gradient in the Kelvin body force field, possibly promoting the instability. Similar is the situation in the case of critical wave numbers and the same is evident from Figure 3.

4.2. Solution by Regular Perturbation Technique

Since the critical wave number is negligibly small for constant-flux thermal boundary conditions (i.e., $D\Theta = 0$ at $z = 0, 1$), the eigenvalue problem is also solved analytically using regular perturbation technique with wave number *a* as a perturbation parameter. Accordingly, W, Θ and Φ are expanded in powers of a^2 as

$$(W, \Theta, \Phi) = (W_0, \Theta_0, \Phi_0) + a^2 (W_1, \Theta_1, \Phi_1) + \cdots. \tag{35}$$

Substituting Eq.(35) into Eqs. (22)-(24) and also in the boundary conditions, and collecting the terms of zeroth order in (a^2), the following obtained

$$D^4 W_0 = 0 \tag{36}$$

$$D^2 \Theta_0 = -W_0 \tag{37}$$

$$D^2 \Phi_0 = -D\Theta_0 \tag{38}$$

with the corresponding boundary conditions

$$W_0 = DW_0 = 0 = D\Theta_0 = \Phi_0 \quad \text{at } z = 0,\ 1. \tag{39}$$

The solution to the zero-th order equations is found to be

$$W_0 = 0,\ \Theta_0 = 1 \text{ and } \Phi_0 = 0. \tag{40}$$

The first order equations in (a^2) are then

$$D^4 W_1 = R_t (1 + M_1) \tag{41}$$

$$D^2 \Theta_1 = 1 - W_1 \tag{42}$$

$$D^2 \Phi_1 = D\Theta_1 \tag{43}$$

with the boundary conditions

$$W_1 = DW_1 = \Phi_1 = D\Theta_1 = 0. \tag{44}$$

The general solution of Eq. (41) is given by

$$W_1 = c_1 \cosh(\sigma_e z) + c_2 \sinh(\sigma_e z) + c_3 + c_4 z - R_e (1 + M_1) z^2 / 2\sigma_e^2 \tag{45}$$

where the arbitrary constants $c_1 - c_4$ are to be determined using the non-uniform temperature profiles and they are given by

Model 1: Linear temperature profile
Model 2: Inverted parabolic temperature profile
Model 3: Parabolic temperature profile
Model 4: Cubic 1
Model 5: Cubic 2
(i) rigid-rigid boundaries

$$c_1 = \frac{R_e (1 + M_1)(1 + \cosh \sigma_e)}{2\sigma_e^3 \sinh \sigma_e},\ c_2 = -\frac{R_e (1 + M_1)}{2\sigma_e^3},\ c_3 = -c_1,\ c_4 = -\sigma_e c_2. \tag{46}$$

(ii) rigid-free boundaries

$$c_1 = -\frac{R_e (1 + M_1)[(\sigma_e^2 - 2)\sinh \sigma_e + 2\sigma_e]}{2\sigma_e^4 (\sinh \sigma_e - \sigma_e \cosh \sigma_e)},\ c_2 = \frac{R_e (1 + M_1)[(\sigma_e^2 - 2)\cosh \sigma_e + 2]}{2\sigma_e^4 (\sinh \sigma_e - \sigma_e \cosh \sigma_e)}$$

$$c_3 = -c_1,\ c_4 = -\sigma_e c_2. \tag{47}$$

Integrating Eq. (45) between z = 0 and z = 1 and using the boundary condition on temperature, it follows that

$$1 = \int_0^1 W_1 dz. \tag{48}$$

Substituting for W_1 from Eq. (37) into Eq. (40) and carrying out the integration leads to an expression for the critical Rayleigh number for rigid-rigid and rigid-free boundaries, respectively, in the form

$$R_{ec}=\frac{12\sigma_e^4\sinh\sigma_e}{(1+M_1)[\sigma_e^2\sinh\sigma_e+12\sinh\sigma_e-6\sigma_e(1+\sinh\sigma_e)]} \tag{49}$$

and

$$R_{ec}=\frac{12\sigma_e^5(\sinh\sigma_e-\sigma_e\cosh\sigma_e)}{(1+M_1)[4\sigma_e(\sigma_e^2-6)\sinh\sigma_e+(24-\sigma_e^4)\cosh\sigma_e+12(\sigma_e^2-2)]}. \tag{50}$$

It is interesting to check the above relations for some special cases. When $M_1=0$ (i.e., ordinary viscous fluid case) then the above equations coincide with those obtained by Shivakumara and Nanjundappa [30].

Letting $\Lambda=1$ and $\sigma_e\to 0$, Eq. (50) becomes

$$\mathrm{R}_c=\frac{720}{1+M_1}. \tag{51}$$

and Eq. (50) becomes

$$R_c=\frac{320}{1+M_1}. \tag{52}$$

thus recovering the results for non-porous case discussed by Nanjundappa and Shivakumara [17]. Noticedthat the above two equations respectively reduce to $R_c=720$ and $R_c=320$ when $M_1=0$ which are the known exact values for the ordinary viscous fluid layer [25]. From Eqs (51) and (52), it is interesting to note that the nonlinearity of fluid magnetization (i.e., M_3) has no effect on the onset of convection; a result which is revealed by numerical computations carried out in the previous section. Since at the onset of convection $a_c=0$ (very large wave length), one would expect that M_3 has no effect on the stability of the system. The numerically computed values of R_c for different values of M_1 , Da^{-1} and Λ with $Bi=0$ are compared in Table 2 with those obtained using regular perturbation technique. The results so obtained from simple regular perturbation technique coincide exactly with those obtained from time consuming numerical methods and thus provides a justification for the analytically obtained results for prescribed heat flux conditions (i.e. $Bi=0$). In other words, the solutions obtained analytically are exact. As noticed earlier, increase in the value of Λ , as well as decrease in M_1 is to increase R_c and raises the stability of the system.

The velocity eigenfunction $W(z)$ is presented in Figures 4 and 5 for different values of M_1 and Λ respectively. As can be seen, increase in the value of MFD viscosity parameter Λ (see Figure 4) is to retard the ferrofluid flow and hence their effect is to delay the onset of convection. But increase in the value of magnetic parameter M_1 (see Figure 5) is to speed up the ferrofluid flow and hence its effect is to hasten the onset of ferroconvection in a ferrofluid saturated porous layer.

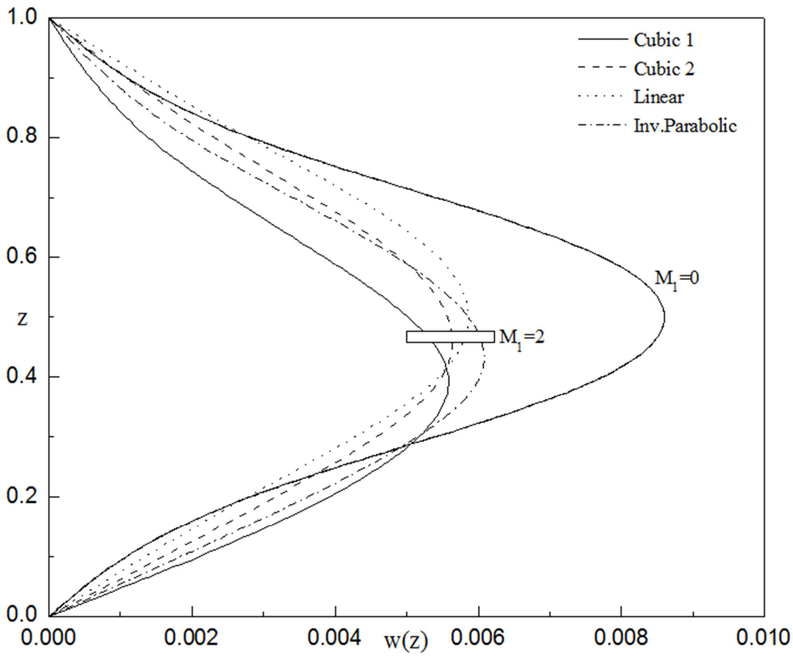

Figure 4. *Velocity Eigen function for two values of* M_1 *when* $\Lambda=0.2$ *and* $Bi=0$.

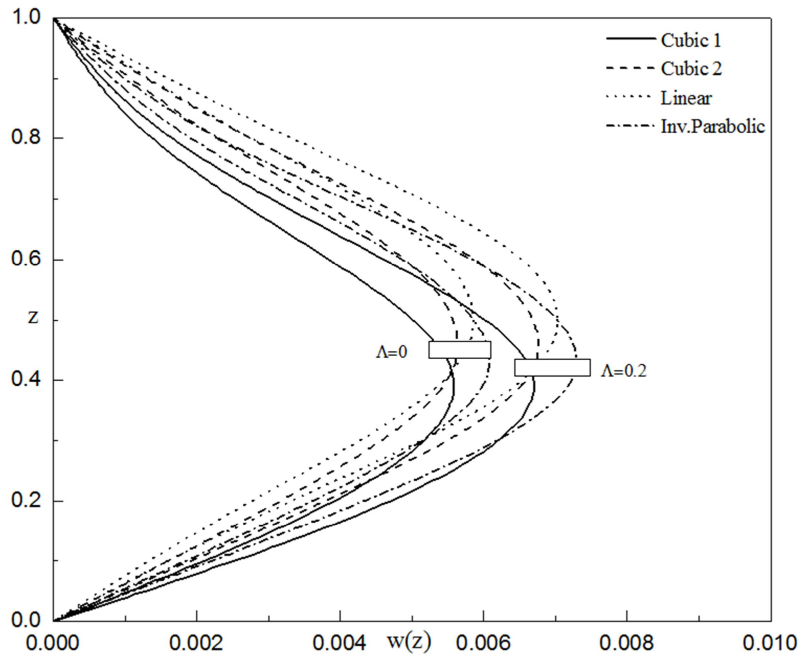

Figure 5. *Velocity Eigen function for two values of* Λ *when* $M_1 = 2$ *and* $Bi = 0$.

5. Conclusions

The linear stability theory is used to investigate the effect of MFD viscosity and non-uniform temperature gradients on the onset ferroconvection in a ferrofluid layer. The lower and upper boundaries of the porous layer are assumed to be rigid by prescribing uniform heat flux condition at the lower boundary and a general thermal condition at the upper boundary. The resulting eigenvalue problem is solved numerically by employing the Rayleigh-Ritz technique.

The following conclusions can be drawn from the present study:

Theeffect of increase in the value of magnetic field dependent viscosity parameter Λ and the Biot number Bi is to delay the onset of ferroconvection, while increase in the value of magnetic number M_1 and nonlinearity of magnetization parameter M_3 is to advance the onset of ferroconvection.

The parameter M_3 has no effect on the stability of the system in the case of constant-flux thermal boundary conditions (i.e., Bi=0).

The numerically and analytically obtained results for the case of constant-flux thermal boundary conditions coincide with each other indicating that the critical stability parameters obtained from the analytical formula are exact.

Nomenclature

$A = (\rho_0 C)_1 / (\rho_0 C)_2$	Ratio of heat capacities (dimensionless)
$a = \sqrt{\ell^2 + m^2}$	Overall horizontal wave number (m)
$\vec{B}$	Magnetic induction field (T)
$Bi = h_t\, d / k_t$	Biot number (dimensionless)
$B = \gamma \beta d$	Viscosity parameter (dimensionless)
C	Specific heat (J / kgK)
$C_{V,H}$	Specific heat at constant volume and magnetic field
d	Thickness of the porous layer
$D = d / dz$	Differential operator
$D / Dt = \partial / \partial t + \vec{q} \cdot \nabla$	Convective derivative
$\underset{\sim}{D} = [\nabla \vec{q} + (\nabla \vec{q})^T] / 2$	Rate of strain tensor
$Da = k / d^2$	Darcy number (dimensionless)
$\vec{g}$	Acceleration due to gravity (m^2 / s)
$\vec{H}$	Magnetic field intensity (A / m)

$H=\left\|\vec{H}\right\|$	Magnitude of $\vec{H}$ (A/m)
H_0	Constant applied magnetic field (A/m)
k	Permeability of the porous medium
k_t	Thermal conductivity (W/mK)
$K=-(\partial M/\partial T)_{H_0,T_0}$	Pyromagnetic coefficient
$\vec{M}$	Magnetization (A/m)
$\vec{M}'=(M'_x, M'_y, M'_z)$	Perturbed magnetization (A/m)
$M=\left\|\vec{M}\right\|$	Magnitude of $\vec{M}$ (A/m)
$M_0=M(H_0,T_0)$	Constant mean value of magnetization (A/m)
$M_1=\Delta T\,K^2\,\mu_0/\rho_0\,\alpha_t\,g\,(1+\chi)d$	Magnetic number (dimensionless)
$M_2=T_0\,K^2\,\mu_0/\rho_0\,C_0\,(1+\chi)$	Magnetic parameter (dimensionless)
$M_3=(1+M_0/H_0)/1+\chi$	Nonlinearity of magnetization parameter
p	Pressure (Pa)
$Pr=\nu/\kappa$	Prandtl number (dimensionless)
$\vec{q}=(u,v,w)$	Velocity vector (m/s)
$R_t=\alpha_t\,g\,\Delta T\,d^3/\nu\kappa$	Thermal Rayleigh number (dimensionless)
$R_d=\alpha_t\,g\,Da^{-1}\Delta T\,d^3/\nu\kappa$	Darcy- Rayleigh number (dimensionless)
t	Time (s)
T	Temperature (K)
T_0	Uniform temperature at lower boundary (K)
T_1	Uniform temperature at upper boundary (K)
$\overline{T}=(T_0+T_1)/2$	Average temperature (K)
T_∞	Temperature in the bulk of the environment (K)
$\Delta T(=T_0-T_1)$	Temperature difference (K)
α_t	Thermal expansion coefficient (K^{-1})
$\beta=\Delta T/d$	Temperature gradient
$\chi=(\partial M/\partial H)_{H_0,T_0}$	Magnetic susceptibility (dimensionless)
$\vec{\delta}$	Variation coefficient of magnetic field dependent (MFD) viscosity
$\delta=\left\|\vec{\delta}\right\|$	Magnitude of $\vec{\delta}$
ε	Porosity of the porous medium (dimensionless)
$\kappa=k_t/\rho_0 C_0$	Thermal diffusivity (m^2/s)
$\Lambda=\dfrac{\tilde{\mu}}{\mu}$	Ratio of viscosities (dimensionless)
$\lambda=\delta\,\mu_0(M_0+H_0)$	Magnetic field dependent (MFD) viscosityparameter ($N\,s/m^2$)
μ	Dynamic viscosity ($Pa\ s$)
$\tilde{\mu}$	Effective dynamic viscosity ($Pa\ s$)
μ_0	Magnetic permeability of vacuum ($=4\pi\times10^{-7}\,N/A^2$)
$\nu=\mu/\rho_0$	Kinematic viscosity (m^2/s)
Φ	Amplitude of perturbed magnetic potential
ρ	Density (kg/m^3)
ρ_0	Reference density at T_0

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