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Comparison of the Solutions of Singular Flierl Petviashvili and the Lane Emden Equations Via Analytical Techniques

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Abstract: The initiative of this paper is to inaugurate a productive comparison of two techniques Adomians Decomposition Method (ADM) and Variational Iteration Method (VIM) for solving the volterra integro-differential equation form of Flierl Petviashvili (FP) and Lane-Emden (LE) equations. The study gives the notable aspects of the two methods and examines the theoretical supporting framework of Adomians Decomposition Method (ADM) and Compares it with Variational Iteration Method (VIM). The aim of the present attempts to find the computational benefits of both the Adomians Decomposition Method (ADM) and Variational Iteration Method (VIM) for solving physical models of FP and LE equations of first and second kind. The Adomians Decomposition method is shown to easily solve the FP and LE equations for all values of the coefficient α, the decomposition series is used to deal with highly nonlinear terms. The Variational Iteration Method also effectively solved the FP and LE equations of first and second kind, both techniques overcome the singular behavior of equations. But after computational results the Variational Iteration Method show impractical approach due to use of Lagrange Multipliers which are not helpful for generalized case. The Adomians Decomposition method shows more efficiency towards generalized FP and LE equations and applications of these equations describes variety of phenomena's in fluid mechanics, relativistic mechanics and in chemically reacting system. The conclusions are supported by numerical calculations of these equations.

Keywords: Flierl-Petviashvili (FP) Equation, Lane-Emden (LE) equation; Variational Iteration Method (VIM), Adomians Decomposition Method (ADM)

1. Introduction

A singular perturbation problem is a well-defined problem in which no single asymptote expansion is consistently authentic at full length of the interval. In the field of Fluid Dynamics, Elasticity, Plasma Dynamics, Magneto Hydrodynamics singular perturbation problems originate periodically. A few striking problems includes Flierl-Petviashvili (FP) equation and Lane-Emden (LE) equations of singular type which are numerically treated in this paper. The numerical treatment of singular nonlinear type of equations faces many computational problems due to singular behavior. There are many analytical techniques are not applicable for singular perturbation problems. Thus more adequate techniques are required to solve singular problems. Kadalbajoo and Reddy [1] and Kadalbajoo and Patidar [2] gives detailed description about the behavior of singular perturbation problems and their analytical treatment [3].

The Flierl-Petviashvili (FP) equation treated by Flierl [5] and then solved by Boyd [6]. The FP equation is basically emerging in diverse conditions for example huge red spot on Jupiter, also for reduction to quasi-geostrophic and for finite height of amplitude redesigning [6]. FP equation has been utilized as the model of inaccessible vortex for Jupiter's airspace and also in geophysical fluids. The FP equation was numerically organized by Flierl [5] and solved using approximating techniques by Boyd [7]. A numerical technique Differential Transformation method was used to solve FP equation by Wazwaz [8]. The FP equation was investigated by Wazwaz in detail by using Adomians Decomposition method with the help of Pade approximation [7] and no exact solution established in literature for this equation. The singularity at $x = 0$, is basically the only difficulty in handling this equation. Many techniques were used for the solution of aforementioned equations for example, Adomians Decomposition and Variational Iteration Method with the combination of Pade Approximation [8, 9].

The general form of FP equation with boundary conditions is given as

$$y'' + \frac{1}{x}y' - y - y^2 = 0, y(0) = \alpha, y'(0) = 0, y(\infty) = 0. \quad (1)$$

Astrophysicists Jonathan Homer Lane and Robert Emden were the first ones who deliberate Lane-Emden equation [10]. LE equation presents the model of density distribution in a gas sphere. Scientist who worked on this equation contemplates the thermal behavior of spherical cloud with mutual attraction of its molecules and subject to the classical law of thermodynamics [11-12]. Lane-Emden equation shows singular behavior which is the only difficulty for handling this equation. The well-known Lane-Emden has been used to model different phenomena's in mathematical physics and astrophysics such as the theory thermionic currents [13, 14]. LE equation of the first kind is useful for computing the structure of interior polytrophic stars. These equations are considerably suitable for analytical and approximate treatment. Many analytical techniques have been used on LE equations such as Adomians Decomposition method, Modified Adomians Decomposition method, Legendre Pseudo spectral approach and Homotopy Perturbation method [15]. The objective of the present study to develop a comparison between two techniques for solving efficiency. For applying VIM on LE equations the standard form of LE equations will be converted into equivalent Volterra Integro Differential Equations [16].

The standard form of Lane-Emden equation is given below

$$y'' + \frac{1}{x}y' + e^y = 0, y(0) = \beta, y'(0) = 0, y(\infty) = 0. \quad (2)$$

The standard form of equation (1)-(2) can be set as

$$y'' + \frac{1}{x}y' + f(y) = 0, y(0) = \alpha. \quad (3)$$

Our objective is to transform above equation in equivalent Volterra –Integro differential equation and then apply Variational Iteration technique and ADM as well. In the present study Lagrange Multipliers use according to the order of these equations. Adomians Decomposition method and variation iteration method are most often used systematic techniques, gives us suitable approximate and exact solution if exist.

Conversion of standard equations into equivalent Volterra Integro-Differential form

Volterra integro differential equations arise in many scientific fields such as wind ripple in the dessert, nono-hydrodynamics, drop wise consideration, glass farming process [4]. These equations are of much importance as many mathematical models expressed in the form of Integro Differential Equations. Essential derivation of standard form of FP and LE equations into equivalent Volterra Integro Differential equation on which techniques will directly applicable is given as follows.

The general form of Eq. (3) with shape factor $k \geq 1$

$$y'' + \frac{k}{x}y' + f(y) = 0, y(0) = \alpha \; k \geq 1 \quad (4)$$

For conversion of above equation into Integro-differential equation,

$$y(x) = \alpha - \frac{1}{k-1}\int_0^x t(1 - \frac{t^{k-1}}{x^{k-1}})f\big(y(t)\big)dt. \quad (5)$$

Differentiating Eq. (5) twice by using Leibniz Rule for differentiation, one can get IEqs. (6) and (7) are

$$y'(x) = -\int_0^x (\frac{t^k}{x^k})f\big(y(t)\big)\,dt, \quad (6)$$

$$y''(x) = -f\big(y(x)\big) + \int_0^x k(\frac{t^k}{x^{k+1}})f\big(y(t)\big)\,dt. \quad (7)$$

Finally Volterra Integro-Differential equation from Lane-Emden equation is

$$y'(x) = -\int_0^x (\frac{t^k}{x^k})f\big(y(t)\big)\,dt \; k \geq 1, y'(0) = \alpha. \quad (8)$$

The solution of these equations graphically compared below to establish effective comparison by showing differences and similarities.

2. Analysis of Techniques

Adomians Decomposition Method (ADM)

Adomians has dispensed and flourished a decomposition method for providing solution of linear or nonlinear problems such as ODEs, PDEs, and IEqs etc. [17]. It works according to the principal of breaking equation into linear and nonlinear parts [18-19]. Reversing the highest-order derivative operator accommodates in the linear operator on both sides, distinguishing the initial or boundary conditions and the terms including the independent variable single as initial approximation, decomposing the unknown function in a series form whose components are to be computed, decomposing the nonlinear function in terms of special polynomials called Adomian polynomials [20-23].

Considering the nonlinear integral equation

$$Ly + Ny = f(t). \quad (9)$$

Applying $L^{-1} = \int_o^x dt$ on both sides of Eq. (9), Eq. (10) is given as

$$y(x) = g(t) - \int_0^x F\big(y(t)\big)dt. \quad (10)$$

Adomians decomposition method defines the unknown function *y(x)* by an infinite series, given as Eq. (11)

$$y(x) = \sum_{n=0}^{\infty} y(x) = y_n(x), \quad (11)$$

where the components $y_n(x)$ are usually computed by recursive relation. The non-linear operator *F(y)* can be decomposed into an infinite series of polynomials given by Eq. (12)

$$F(y) = \sum_{n=0}^{\infty} A_n, \quad (12)$$

where A_n are also-called Adomian polynomials of $y_0, y_1, y_2, \ldots, y_n$ define by

$$A_n = \frac{1}{n!}\frac{d^n}{d\ ^n}\left[F(\sum_{i=0}^{n}\ ^i y_i)\right]_{\ =0}, \text{ n= 0, 1, 2, 3…} \quad (13)$$

Or equivalently

$$A_0 = F(y_0),$$

$$A_1 = y_1 F'(y_0),$$

$$A_2 = y_2 F'(y_0) + \frac{1}{2} y_1^2 F''(y_0),$$

$$\vdots$$

The recursive relation by using equation (9) and (10) is given as

$$y_0(x) = g(t),$$

$$\vdots$$

$$y_{n+1} = -\int_0^x \int_0^x (\frac{t^k}{x^k}) f(y_n(t))\, dtdt, k \geq 1, n \geq 0. \quad (14)$$

These components can be computed for all types of non-linear equations by using this analysis. Many techniques has been developed recently by Wazwaz [23] established effective polynomials. These polynomials will help to calculate non-linear terms in a much easier way with less calculations.

Variational Iteration Method (VIM)

Let us assume a differential equation of the form

$$Ly + Ny = g(t), \quad (15)$$

In above equation *L* represents linear operator of higher order and *N* represents non-linear operator, *g(t)* is source term. This term denote the inhomogeneous part.

To apply VIM on equation above, a correction functional is established as follows [24-25]

$$y_{n+1}(t) = y_n(t) + \int_0^x \lambda(Ly_n(\xi) + Ny_n(\xi) - g(\xi))d\xi, \quad (16)$$

Where λ is Lagrange multiplier, its value can be calculated optimally by using variational theory. y_n is restricted variation and $\delta y_n = 0$.

The approximate value of Lagrange multiplier depends on orders of derivatives. In present analysis, we will use only value of Lagrange multiplier as $(\xi) = -1$; initial approximation will be obtained by initial conditions [26-28]. Successive approximations depends on initial guess for $n \geq 0$.

For equation the correction functional will be

$$y_{n+1} = y_n(x) - \int_0^x (y_n'(t) + \int_0^t \frac{r}{t} f(r)dr)\, dt, y(0) = \alpha. \quad (17)$$

3. Flierl-Petviashvili (FP) Integral Equation (Solution by VIM)

$$y'(x) = -\int_0^x \frac{t}{x}(-y(t) - y^2(t))dt, y(0) = \alpha, y'(0) = 0, y(\infty) = 0. \quad (18)$$

The correction functional for Eq. (17) can be written as

$$y_{n+1}(x) = y_n(x) - \int_0^x (y_n'(x) + \int_0^t \frac{r}{t}(-y_n(r) - y_n^2(r))dr)dt, y(0) = \alpha, n \geq 0. \quad (19)$$

Initial approximation is given as

$$y_0(x) = \alpha.$$

First approximation for *n*=0

$$y_1(x) = \alpha - \int_0^x (y_0'(t) + \int_0^t \frac{r}{t}(-y_0(r) - y_0^2(r))dr)\, dt,$$

$$y_1(x) = \alpha + \frac{\alpha(\alpha+1)}{4} x^2,$$

2nd approximation for n=1

$$y_2(x) = \alpha + \frac{\alpha(1+\alpha)x^2}{4} - \int_0^x (y_1'(t) + \int_0^t \frac{r}{t}(-y_1(r) - y_1^2(r))dr)dt,$$

$$y_2(x) = \alpha + \frac{\alpha(\alpha+1)}{4}x^2 + \frac{\alpha(\alpha+1)(2\alpha+1)}{64}x^4 + \frac{\alpha^2(\alpha+1)^2}{576}x^6,$$

3rd approximation for n=2

$$y_3(x) = \alpha + \frac{\alpha(\alpha+1)}{4}x^2 + \frac{\alpha(\alpha+1)(2\alpha+1)}{64}x^4 + \frac{\alpha^2(\alpha+1)^2}{576}x^6 - \int_0^x y_2'(t) + \int_0^t \frac{r}{t}\left(-y_2(r) - y_2^2(r)\right)dr)dt,$$

$$y_3(x) = \alpha + \frac{\alpha(\alpha+1)}{4}x^2 + \frac{\alpha(\alpha+1)(2\alpha+1)}{64}x^4 + \frac{\alpha(\alpha+1)(8\alpha^2+8\alpha+1)}{2304}x^6, + \frac{11\alpha^2(\alpha+1)^2(2\alpha+1)}{73728}x^8 + \cdots + O(x^{10}),$$

$$\vdots,$$

Graph of the final iteration is given as

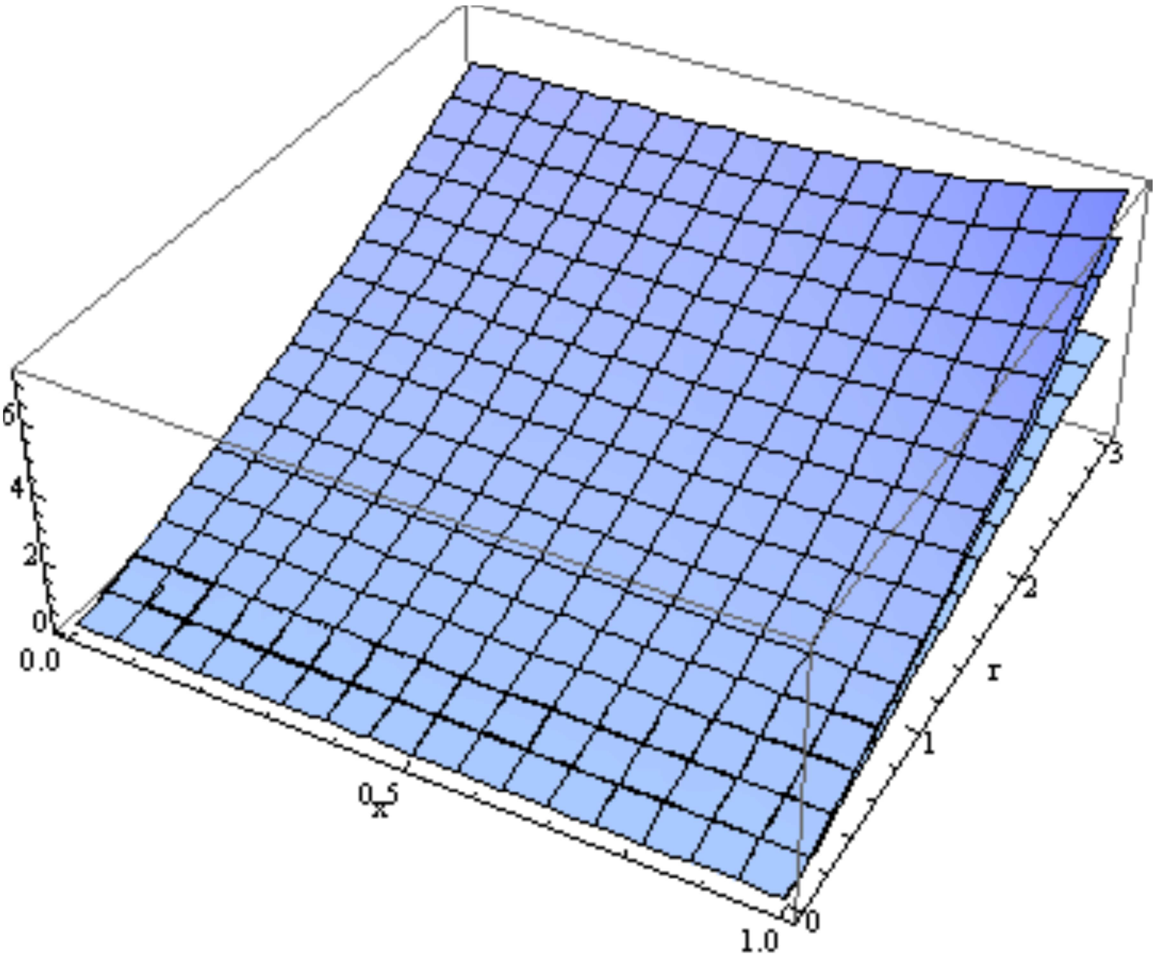

Figure 1. *Show the final iteration of variation iteration method for positive value of α.*

The solution of FP equation graphically gives layers of surfaces because in Variational Iteration Method (VIM) the correction functional represents a complete solution in all upcoming iterations. Each upcoming iteration contains all previous iterations and not dependent on any solution series. Variational Iteration Method always gives rapid solutions with the help of Lagrange Multipliers which are not easily calculated in all cases of non-linearity and singularity. It can be observed from the above iteration that the upcoming approximations will be consisting of large number of terms. The convergence point is abundantly very large. Power series method is not useful for BVPs so, to obtain better solution, we use *Pade Approximation* with boundary condition $y(\infty) = 0$ but the convergence of approximants increases by order. So, the solution of this equation exists in series form.

Flierl-Petviashvili (FP) Integral Equation (solution by ADM)

As Eq. (17)

$$y'(x) = -\int_0^x \frac{t}{x}\left(-y(t) - y^2(t)\right)dt, y(0) = \alpha, y'(0) = 0, y(\infty) = 0. \tag{20}$$

Integrating Eq. (17) to apply ADM and using $y(0) = \alpha$ we get,

$$y(x) = \alpha - \int_0^x \int_0^x \frac{t}{x}\left(-y(t) - y^2(t)\right)dtdt, \tag{21}$$

$$y(x) = \alpha + \frac{1}{x}\int_0^x \int_o^x ty(t)dtdt + \frac{1}{x}\int_0^x \int_o^x y^2(t)dtdt, \tag{22}$$

according to ADM

Let $y(x) = \sum_0^\infty y_n(x)$

$put\ in\ (21)\ and$ obtain Eq. (23)

1st approximation

$$y_0(x) = \alpha$$

Recursive relation

$$y_{n+1}(x) = \frac{1}{x}\int_0^x \int_o^x ty_n(t)dtdt + \frac{1}{x}\int_0^x \int_o^x ty_n^2(t)dtdt\,, for\ n \geq 0, \tag{23}$$

2nd approximation for n=0

$$y_1(x) = \frac{1}{x}\int_0^x \int_o^x t\alpha\ dtdt + \frac{1}{x}\int_0^x \int_o^x t\alpha^2\ dtdt,$$

$$y_1(x) = \frac{\alpha(1+\alpha)}{2}x^2,$$

3rd approximation for n=1

$$y_2(x) = \frac{1}{x}\int_0^x \int_o^x ty_1(t)dtdt + \frac{1}{x}\int_0^x \int_o^x ty_1^2(t)dtdt,$$

(by using Adomians polynomials for nonlinear term)

$$y_2(x) = \frac{\alpha(\alpha+1)(2\alpha+1)}{8}x^4,$$

4th approximation for n=2

$$y_3(x) = \frac{1}{x}\int_0^x \int_o^x ty_2(t)dtdt + \frac{1}{x}\int_0^x \int_o^x ty_2^2(t)dtdt,$$

$$y_3(x) = \frac{\alpha(\alpha+1)(6\alpha^2+6\alpha+1)}{48}x^6,$$

$$\vdots,$$

now substituting these approximations in above series solution, we get

$$y(x) = \alpha + \frac{\alpha(\alpha+1)}{2}x^2 + \frac{\alpha(\alpha+1)(2\alpha+1)}{8}x^4 + \frac{\alpha(\alpha+1)(6\alpha^2+6\alpha+1)}{48}x^6 + \cdots$$

Graphically this solution can be observed as

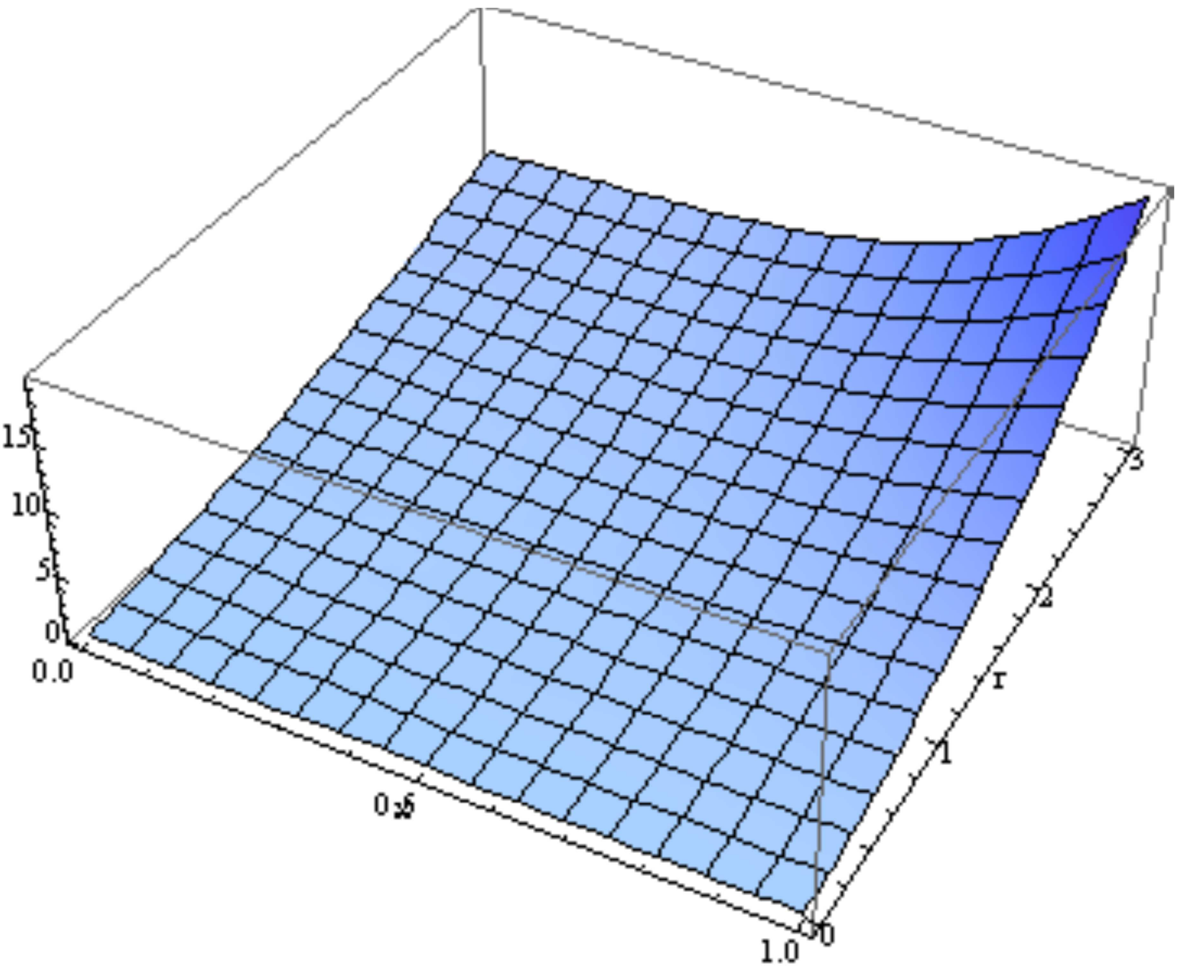

Figure 2. *Show the final iteration of ADM for positive value of α.*

We again calculated a series form solution with a different technique but with a slight difference of coefficients with the parameter α. So, we can conclude a statement by comparing the results obtained by both techniques that is the solution of this type of FP equation with these boundary conditions is always converges to large amount of terms and is approximate solution always.

4. The Lane Emden Equation of the Second Kind (Solution by VIM)

$$y"(x) + \frac{1}{x}y' + 2e^{y} = 0, y(0) = \beta, y'(0) = 0, y(1) = 0. \quad (24)$$

As k=1 so, by generalized equation

$$y'(x) = -\int_0^x \frac{t}{x} 2e^{y(t)} dt \; y(0) = \beta, \quad (25)$$

According to VIM correction functional will be

$$y_{n+1}(x) = y_n(x) + \lambda \int_0^x (y_n'(t) + \int_0^t \frac{r}{t}(2e^{y_n(r)})dr)dt), \quad (26)$$

here, $\lambda = -1$

Initial approximation will be

$$y_0(x) = \beta,$$

2nd approximation for *n=0*

$$y_1(x) = \beta - \int_0^x (0 + \int_0^t \frac{r}{t}(2e^{\beta})dr)dt,$$

$$y_1(x) = \beta - \frac{1}{2}x^2 e^{\beta},$$

3rd approximation for *n=1*

$$y_2(x) = y_1(x) - \int_0^x (y_1'(t) + \int_0^t \frac{r}{t}\left(2e^{y_1(r)}\right)dr)dt,$$

$$y_2(x) = \beta - \frac{1}{2}x^2 e^{\beta} + \frac{1}{16}x^4 e^{2\beta},$$

4th approximation for *n=2*

$$y_3(x) = y_2(x) - \int_0^x (y_2'(t) + \int_0^t \frac{r}{t}\left(2e^{y_2(r)}\right)dr)dt,$$

$$y_3(x) = \beta - \frac{1}{2}x^2 e^{\beta} + \frac{1}{16}x^4 e^{2\beta} - \frac{1}{144}e^{3\beta}x^6 + \frac{1}{1536}e^{4\beta}x^8 - \frac{1}{19200}e^{5\beta}x^{10} +,$$

$$\vdots,$$

By using the boundary condition $y(1) = 0$ in the following approximations, we get value of $\beta = 1.386294260$ which will convert the approximations in closed form

Given by

$$y(x) = 1.386294260 - 2x^2 + x^4 - \frac{2}{3}x^6 + \frac{1}{3}x^8 - \frac{2}{5}x^{10} + \cdots.$$

Turn into exact solution of the form

$$y(x) = 2\ln\left(\frac{2}{(x^2+1)}\right),$$

The Lane Emden Equation of the Second Kind (Solution By ADM)

$$y''(x) + \tfrac{1}{x}y' + 2e^y = 0, y(0) = \beta, y'(0) = 0, y(1) = 0, \tag{27}$$

as k=1 so, by generalized equation

$$y'(x) = -\int_0^x \tfrac{t}{x} 2e^{y(t)}dt\; y(0) = \beta, \tag{28}$$

Integrating Eq. (28) to apply ADM, Eq. (29) will be

$$y(x) = \beta - \int_0^x \int_0^x \tfrac{t}{x} 2\left(e^{y(t)}\right)dtdt, \tag{29}$$

According to ADM

Let $y(x) = \sum_0^\infty y_n(x)\; put\; in\; (28) and$

1st approximation

$$y_0(x) = \beta.$$

Recursive relation

$$y_{n+1}(x) = -\int_0^x \int_0^x \tfrac{t}{x} 2\left(e^{y_n(t)}\right)dtdt, \tag{30}$$

(here we are using Adomians Polynomials for non-linear terms)

2nd approximation for *n=0*

$$y_1(x) = -\int_0^x \int_0^x \frac{t}{x} 2\left(e^{y_0(t)}\right)dtdt,$$

$$y_1(x) = -e^{\beta}x^2,$$

3rd approximation for *n=1*

$$y_2(x) = -\int_0^x \int_0^x \frac{t}{x} 2\left(e^{y_1(t)}\right)dtdt,$$

$$y_2(x) = \frac{1}{2}e^{2\beta}x^4,$$

3rd approximation for *n=2*

$$y_3(x) = -\int_0^x \int_0^x \frac{t}{x} 2\left(e^{y_2(t)}\right)dtdt,$$

$$y_3(x) = -\frac{1}{16}e^{3\beta}x^6,$$

$$\vdots,$$

Substituting these components into solution series, we get

$$y(x) = \beta - x^2 e^{\beta} + \frac{1}{4}x^4 e^{2\beta} - \frac{1}{16}e^{3\beta}x^6 + \frac{1}{144}e^{4\beta}x^8 - \frac{1}{1536}e^{5\beta}x^{10} + \cdots,$$

and exact solution will be

$$y(x) = \ln\left(\frac{1}{(x^2+1)}\right),$$

5. Conclusion

The aim of this article to obtaine the solution of LE and FP equations via Adomians Decomposition method and Variational Iteration method. The present study show that VIM suffers when applied on LE and FP equations with the help of approximate Lagrange multipliers which are not helpful for generalized case. On the other hand ADM provides a new algorithm for solving these equations for their generalized equations, with the help of Adomians polynomials for nonlinearity, although ADM gives solutions after long calculations but this technique is more approachable for all cases.

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