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Non-Lipschitz Sobolev Type Fractional Neutral Impulsive Stochastic Differential Equations with Fractional Stochastic Nonlocal Condition, Infinite Delay and Poisson Jumps

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Abstract: In the present paper, we have established the existence and uniqueness of mild solutions for non-Lipschitz Sobolev type fractional neutral impulsive stochastic differential equations satisfying fractional stochastic nonlocal condition with infinite delay and Poisson jumps in $\mathcal{L}_p$ space. we adopt the non-Lipschitz condition proposed by Taniguchi (2009) which is a much weaker condition with wider range of applications. The existence of mild solutions is established by using strong and powerful tool called Picard's approximation technique. We can find that the similar existence results are suitable for those non-Lipschitz Sobolev type fractional neutral impulsive stochastic differential equations satisfying fractional stochastic nonlocal condition of different fractional orders with infinite delay and Poisson jumps in $\mathcal{L}_p$ space. At the end an example is given to illustrate the theory.

Keywords: Stochastic Differential Equations of Sobolev-Type, Existence and Uniqueness, Fractional Derivatives, Non-Lipschitz Coefficients

1. Introduction

Fractional differential equations are well-known to describe many sophisticated dynamical systems in many applied fields such as electrochemistry control, seepage flow in porous media, fluid dynamics, viscoelsticity, traffic models, electro magnetic and engineering. The most important advantages of using fractional differential equations in these and other applications is their nonlocal property. The nonlocal condition, in many cases, has a better effect than the usual classical condition. This means that the next state of a system depends not only upon its current state of a system, but also upon all its historical states. Besides, noise or stochastic perturbation is unavoidable in nature as well as in man made systems. Hence it is of great significance to import stochastic effect into the investigation of fractional differential systems. Various evolutionary processes from fields such as population dynamics, aeronautics, economics and engineering are characterized by the fact that they undergo abrupt changes of state at certian moments of time between intervals of continuous evolution. Because the duration of these changes are often negligible compared to the total duration of the process, such changes can be reasonably well approximated as being instantaneous changes of state, or in the form of impulses. The impulsive differential equations arising from the real world problems to describe the dynamical process subject to a great changes in short times issued, for instance, in biotechnology, automatics, population dynamics, economics, electrical engineering, drug administration, threshold theory in biology, ecology and robotics etc., refer [1-5]. Neutral differential equations aries in many areas of applied mathematics and such equations have received much attention in recent years. Neutral differential systems with impulses aries in many areas of applied mathematics and these systems have been studied during the last decades. The existence result of fractional neutral stochastic integro-differential equations with infinite delay is obtained in [6] under non-Lipschitz conditions by means of Sadovskii's fixed point theorem. Very recently, controllability of nonlocal second-order impulsive neutral stochastic functional integro-differential equations with delay and Poisson jumps have attracted in interest [7].

Nonlinear integer order differential systems was studied by many authors. In modern years, there has been significant development in fractional differential equations including monographs of Miller and Ross [8], Podlubny [9] and the papers. A new set of necessary and sufficient conditions are formulated for the approximate controllability of fractional stochastic neutral integro-differential inclusions with infinite delay have received much attention [10]. Further it is important to note that event driven dynamics become very useful in most fields of application and lead to stochastic differential equations with jumps [11-14]. However, to the best of our knowledge, the neutral impulsive fractional stochastic differential equations satisfying nonlocal conditions with infinite delay and poisson jumps arise in many areas of applied mathematics [15].

The Sobolev type semilinear integrodifferential equation serves as an abstract formulation of partial integrodifferential equation, which arises in many fields such as in the flow of fluid through fissured rocks [16], thermodynamics and shear in second order fluids, in soil mechanics and propagation of long waves of small amplitude. Also the mathematical modeling and simulations of systems and processes are based on the description of their properties in terms of fractional integrodifferential equation of Sobolev type [17]. Some authors investigated the approximate boundary controllability of Sobolev-type stochastic differential systems in Hilbert space [18]. In the previous work some people have introduced a more appropriate concept of the existence of mild solutions for fractional integrodifferential equations of Sobolev type with nonlocal conditions [19]. Controllability of Sobolev type nonlocal impulsive mixed functional integrodifferential evolution systems has been studied by Kumar and Kumar [20]. Therefore, it is of great significance to import the stochastic effects into the investigation of existence and controllability results for Sobolev-type fractional impulsive differential equations with infinite delay [21]. A new notion called fractional stochastic nonlocal condition, and the approximate controllability of a class of fractional stochastic nonlinear differential equations of Sobolev type in Hilbert spaces has been introduced in [22]. Approximate controllability of fractional Sobolev type stochastic differential equations driven by mixed fractional Brownian motion has been established by Abid et al. [23].

In this paper, we show the existence and uniqueness results for (1) by means of the Picard type approximation technique. Compared with the earlier related existence results that appeared in [24-26], the existence results of Sobolev-type stochastic differential equations with non-Lipschitz coefficients have been investigated by many researchers ([27] and references therein). The existence and uniqueness of mild solutions to non-Lipschitz stochastic neutral delay evolution equation driven by Poisson jump processes has been described by Luo and Taniguchi [28]. The fractional evolution equations with infinite delay is studied in [29] under Caratheôdory conditions. The Carathéodory approximations and stability of solutions to non-Lipschitz stochastic fractional differential equations of Itô-Doob type is reported in [30]. Some authors established the sufficient conditions of existing results of stochastic differential system with infinite delay in $\mathcal{L}_{\mathrm{p}}$ space [31, 32]. Motivated by the previous mentioned problem, our current consideration is on the existence and uniqueness of mild solutions for non-Lipschitz Sobolev type fractional neutral impulsive stochastic differential equations satisfying fractional stochastic nonlocal condition with infinite delay and Poisson jumps in $\mathcal{L}_{\mathrm{p}}$ space. The purpose of this paper is to study the existence and uniqueness of mild solutions for non-Lipschitz Sobolev type fractional neutral impulsive stochastic differential equations satisfying fractional stochastic nonlocal condition with infinite delay and Poisson jumps in $\mathcal{L}_{\mathrm{p}}$ space using Picard type approximation technique described by

$$
\begin{aligned}
{}^{c}D_t^q[Vx(t)-g(t,x_t)] &= [Ax(t)+f(t,x_t)]dt+\sigma_1(t,x_t)\frac{d\omega_1(t)}{dt}+\int_Z \mathfrak{h}(t,x_t,\eta)\tilde{N}(dt,d\eta),\, t\in J:=[0,b]\setminus\{t_1,\dots,t_n\} \\
{}^{V}D_t^{1-q}[x(t)]|_{t=0} &= \sigma_2(t,x_t)\frac{d\omega_2(t)}{dt} \\
\Delta x(t_k) &= I_k(x_{t_k}),\, k=1,2,\dots,n. \qquad (1)
\end{aligned}
$$

${}^{c}D_t^q$ and ${}^{V}D_t^{1-q}$ are the Caputo and Riemann-Liouville fractional derivatives with $0<q\le 1$. Let X and Y be two Hilbert spaces and let the state $x(\cdot)$ take its values in X. Let us assume that the operators V and A be defined on domains contained in X and ranges contained in Y. $V: D(V)\subset X\to Y$ and $M: D(M)\subset X\to Y$ are linear operators and M is closed. Let $0=t_0<t_1<t_2<\ldots<t_n<b$ be the given time points. Let K be another separable Hilbert space. Let $\{\omega(t)\}_{t\ge 0}$ be a given K-valued Wiener process with a finite trace nuclear covariance operator $Q\ge 0$. Let $\tilde{q}=\{\tilde{q}(t): t\in D_{\tilde{q}}\}$ be a stationary $\mathfrak{F}_t$ Poisson point process with characteristic measure λ. Let $N(dt,d\eta)$ be the Poisson counting measure associated with $\tilde{q}$. Then $N(t,Z)=\sum_{s\in D_{\tilde{q}},s\le t} I_Z(\tilde{q}(s))$ with measurable set $Z\in\tilde{B}(K-\{0\})$, which denotes the Borel σ-field of $K-\{0\}$. Let $\tilde{N}(dt,d\eta)=N(dt,d\eta)-dt\lambda(d\eta)$ be the compensated Poisson measure that is independent of $\omega_1(t)$ and $\omega_2(t)$. Let $P_2([0,b]\times Z;H)$ be the space all mapping $\chi:[0,b]\times Z\to H$ for which $\int_0^b\int_Z \mathbb{E}\,\|\chi(t,\eta)\|_H^{\vartheta}\,\lambda(d\eta)dt<\infty$ for $\vartheta=2$ and $\vartheta=4$. We can define the H-valued stochastic integral $\int_0^b\int_Z \chi(t,\eta)\tilde{N}(dt,d\eta)$, which is a centred square integrable martingale. We can also employing the same notation $\|\cdot\|$ for the norm of $\mathcal{L}(K,H)$, which denotes the space of all bounded operators from K into H. The histories x_t represents the function defined by $x_t:(-\infty,0]\to Y$ such that $x_t(\theta)=x(t+\theta)$, for $t\geq 0$ and $\theta\leq 0$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically. Further $f:J\times X\to Y$, $g:J\times X\to Y$, $\sigma_1:J\times X\to\mathcal{L}_2^0$, $\sigma_2:J\times X\to\mathcal{L}_2^0$ and $\mathfrak{h}:J\times X\times Z\to Y$ are nonlinear functions. Here $\mathcal{L}_Q(K,H)$ denotes the space of all Q-Hilbert Schmidt operators from K into H. Moreover $I_k:X\to Y$ is an appropriate function. The symbol $\Delta\zeta(t)$ represents the jump of the function ζ at t, defined by $\Delta\zeta(t)=\zeta(t^+)-\zeta(t^-)$.

2. Preliminaries

Let $(\Omega,\mathfrak{F},\{\mathfrak{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete probability space equipped with a normal filteration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions (ie., right continuous, $\mathfrak{F}_t\subset\mathfrak{F}$ and $\mathfrak{F}_0$ containing all $\mathbb{P}$-null sets). Let $\mathbb{E}(\cdot)$ denote the expectation with respect to the measure $\mathbb{P}$. We consider three real separable spaces X, Y and $\mathbb{E}$ and Q-Wiener process on $(\Omega,\mathfrak{F},\{\mathfrak{F}_t\}_{t\geq 0},\mathbb{P})$ with the linear bounded covariance operator Q such that $Tr(Q)<\infty$, $Tr(Q)$ denotes the trace of the operator Q. Further, we assume that there exists a complete orthonormal systems $\{e_{1,n}\}_{n\geq 1}$, $\{e_{2,n}\}_{n\geq 1}$ in $\mathbb{E}$, bounded sequences of non-negative real numbers $\{\lambda_{1,n}\}$,$\{\lambda_{2,n}\}$ such that $Qe_{1,n}=\lambda_{1,n}e_{1,n}$ and $Qe_{2,n}=\lambda_{2,n}e_{2,n}$, $n=1,2,...$, and sequences $\{\beta_{1,n}(t)\}_{n\geq 1}$ and $\{\beta_{2,n}(t)\}_{n\geq 1}$ of independent Brownian motions such that

$$\langle\omega_1(t),e_1\rangle=\sum_{n=1}^{\infty}\sqrt{\lambda_{1,n}}\langle e_{1,n},e_1\rangle\beta_{1,n}(t), e_1\in\mathbb{E}, t\in J,$$

$$\langle\omega_2(t),e_2\rangle=\sum_{n=1}^{\infty}\sqrt{\lambda_{2,n}}\langle e_{2,n},e_2\rangle\beta_{2,n}(t), e_2\in\mathbb{E}, t\in J.$$

and $\mathfrak{F}_t=\mathfrak{F}_t^{\omega}$, where $\mathfrak{F}_t^{\omega}$ is the σ-algebra generated by $\{\omega_1(s),\omega_2(s):0\leq s\leq t)\}$. Let $\mathcal{L}_2^0=\mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{E};X)$ be the space of all-Hilbert Schmidt operators from $Q^{\frac{1}{2}}\mathbb{E}$ to X with the norm $\|\varphi\|_{\mathcal{L}_2^0}=Tr(\varphi Q\varphi^{\star})<\infty$, $\varphi\in\mathcal{L}(\mathbb{E},X)$ Let $\mathcal{L}_2(\mathfrak{F}_T,X)$ be the Hilbert space all $\mathfrak{F}_T$-measurable square integrable randon variables with values in the Hilbert space X and $\mathfrak{F}_T=\mathfrak{F}$. Let $\mathcal{L}_2^{\mathfrak{F}}(J,X)$ be the Hilbert space of all square integrable and $\mathfrak{F}_t$-adapted processes with values in the Hilbert space X. Let $\mathcal{C}(J;\mathcal{L}_2(\mathfrak{F},X))$ be the Hilbert space of continuous maps from J into $\mathcal{L}_2(\mathfrak{F},X))$ satisfying $\sup_{t\in J}\mathbb{E}\|X(t)\|^{\mathrm{p}}<\infty$. Let $H_2(J;X)$ be the closed subspace of $\mathcal{C}(J;\mathcal{L}_2(\mathfrak{F},X))$ consisting of a measurable and $\mathfrak{F}_t$-adapted X-valued process $x\in\mathcal{C}(J;\mathcal{L}_2(\mathfrak{F},X))$ endowed with the norm $\|x\|_{H_2}=(\sup_{t\in J}\mathbb{E}\|x(t)\|_X^{\mathrm{p}})^{\frac{1}{\mathrm{p}}}$. Let $\mathcal{B}_T$ denote the Banach space of all X-valued $\mathfrak{F}_t$-adapted processes $x(t,w):J\times\Omega\to X$, which are continuous in t for a. e, fixed $\omega\in\Omega$ and satisfy

$$\|x\|_{\mathcal{B}_T}=\mathbb{E}(\sup_{t\in[0,T]}\|x(t,\omega)\|_X^{\mathrm{p}})^{\frac{1}{\mathrm{p}}}<\infty, \mathrm{p}\geq 2.$$

An X-valued random variable is an $\mathfrak{F}$-measurable function $x(t):\Omega\to X$ and a collection of random variables $S=\{x(t,w):\Omega\to X|_{t\in J}\}$ is called a stochastic process. Usually, we suppress the dependence on $w\in\Omega$ and write $x(t)$ instead of $x(t,w)$ and $x(t):J\to X$ in the place of S.

Definition 2.1 The fractional integral of order q with the lower limit zero for a function $h:\mathbb{R}_+\to\mathbb{R}$ is defined as

$$I^q h(t)=\frac{1}{\Gamma(q)}\int_0^t\frac{h(s)}{(t-s)^{1-q}}ds, t>0, q>0,$$

provided the right side is point-wise defined on $\mathbb{R}_+$, where $\Gamma(\cdot)$ is the gamma function, which is defined as $\Gamma(y)=\int_0^{\infty}t^{y-1}e^{-t}dt$.

Definition 2.2 The Riemann-Liouville derivative of order $n-1<q<n, n\in\mathbb{N}$, for a function $h\in\mathcal{C}^n(\mathbb{R}_+)$ is given by

$${}^lD^q h(t)=\frac{1}{\Gamma(n-q)}\frac{d^n}{dt^n}\int_0^t\frac{h(s)}{(t-s)^{q+1-n}}ds, t>0.$$

Definition 2.3 The Caputo derivative of order $n-1<q<n, n\in\mathbb{N}$, for a function $h\in\mathcal{C}^n(\mathbb{R}_+)$ is given by

$${}^cD^q h(t)=\frac{1}{\Gamma(n-q)}\int_0^t\frac{h^{(n)}(s)}{(t-s)^{q+1-n}}ds=I^{n-q}h^{(n)}(t), t>0.$$

Lemma 2.4 [33, 34] Let the space $M_v^{\theta}([0,T]\times\Omega\times(K-\{0\}),X),(\theta\geq 2)$ denote the set of all random process $\eta(t,y)$ with values in X predictable with respect to $\{\mathfrak{F}_t\}_{t\geq 0}$ such that

$$\mathbb{E}\left(\int_0^T\int_Z\|\eta(t,y)\|_X^{\theta}\,vdydt\right)<\infty.$$

Suppose $\eta\in M_v^2([0,T]\times\Omega\times(K-\{0\}),X)\cap M_v^4([0,T]\times\Omega\times(K-\{0\}),X)$. Then for any $t\in[0,T]$,

$$\mathbb{E}(\sup_{0\leq\tau\leq t}\|\int_0^T\int_Z S(\tau-s)\eta(s,y)v(dyds)\|_X^2)\leq C[\mathbb{E}(\int_0^t\int_Z\|\eta(s,y)\|_X^2\,v(dy)ds)+\mathbb{E}(\int_0^t\int_Z\|\eta(s,y)\|_X^4\,v(dy)ds)^{\frac{1}{2}}]$$

for some number $C=C(T)>0$, dependent on $T>0$.

3. The Main Results

In order to derive the existence and uniqueness of mild solutions of the fractional stochastic system (1), we need to consider the following basic definitions

In this paper we will define an axiomatic definition of the phase space $\mathcal{B}$ denotes the Banach space of functions defined from $(-\infty,0]$ into X endowed with a seminorm denoted as $\|\cdot\|_{\mathcal{B}}$ such that the following axioms hold (refer [35]):

(a) If $x:(-\infty,b]\to X$, $b>0$ is continuous on J and

$x_0 \in \mathcal{B}$, then for each $t \in J$ the following conditions hold:

(i) $x_t \in \mathcal{B}$,

(ii) $\| x(t) \| \leq L_1 \| x_t \|_{\mathcal{B}}$,

(iii) $\| x_t \|_{\mathcal{B}} \leq L_2(t) \| x_0 \|_{\mathcal{B}} + L_3(t)\sup\{\| x(s) \|: 0 \leq s \leq t\}$, where $L_1 > 0$ is a constant, $L_2(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is a locally bounded function, $K_3(\cdot): \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function. Also L_1, $L_2(\cdot)$ and $L_3(\cdot)$ are independent of $x(\cdot)$,

(b) For the function $x(\cdot)$ in (a), x_t is a $\mathcal{B}$-valued continuous function on $[0, b]$,

(c) Tthe space $\mathcal{B}$ is complete.

Definition 3.1 [28] The space $(\mathcal{B}_T, \|\cdot\|_{\mathcal{B}_T})$ denotes the space of all the X-valued and $\mathfrak{F}_t$-adapted processes $x(t)$ defined on $[0, T], T > 0$ such that

$$\| x \|_{\mathcal{B}_T} = \mathbb{E}(\sup_{t\in[0,T]} \| x(t) \|_X^{\mathfrak{p}})^{\frac{1}{\mathfrak{p}}} < \infty, \mathfrak{p} \geq 2.$$

Then it is clear that $\mathcal{B}_T$ is a Banach space.

Lemma 3.2 [36] *For any* $\mathfrak{p} \geq 2$ *and let* $G: J \times \Omega \to \mathcal{L}_2^0$ *be* $\mathcal{L}_2^0$ *-valued predictable process such that* $\mathbb{E}(\int_0^T \| G(r) \|_{\mathcal{L}_2^0}^{\mathfrak{p}} dr) < \infty$, *we have*

$$\mathbb{E}(\sup_{s\in[0,t]} \| \int_0^s G(r)d\omega(r) \|^{\mathfrak{p}}) \leq c_{\mathfrak{p}} \sup_{s\in[0,t]} \mathbb{E}(\| \int_0^s G(r)d\omega(r) \|^{\mathfrak{p}}), \leq C_{\mathfrak{p}}\mathbb{E}(\int_0^t \| G(r) \|_{\mathcal{L}_2^0}^2 dr)^{\frac{\mathfrak{p}}{2}}, t \in [0, T],$$

where $c_{\mathfrak{p}} = (\frac{\mathfrak{p}}{\mathfrak{p}-1})^{\mathfrak{p}}$ and $C_{\mathfrak{p}} = (\frac{\mathfrak{p}}{2}(\mathfrak{p}-1))^{\frac{\mathfrak{p}}{2}}(\frac{\mathfrak{p}}{\mathfrak{p}-1})^{\frac{\mathfrak{p}^2}{2}}$

Definition 3.3 [22, 23, 27, 32, 37, 38] An X-valued stochastic process $x \in H_2(J, X)$ is said to be a mild solution of the system (1) if

(i) $x(t)$ is $\mathfrak{F}_t$-adapted and measurable for $t \geq 0$,

(ii) $x(t)$ is continuous on $[0, b]$ $\mathfrak{p}$-almost surely and for each $s \in [0, t)$,

$$\mathbb{P}(\int_0^t [\| (t-s)^{q-1}T_V(t-s)f(s, x_s) \| + \| (t-s)^{q-1}T_V(t-s)\sigma_1(s, x_s) \|_{\mathcal{L}_2^0} + \| (t-s)^{-q}\sigma_2(s, x_s) \|_{\mathcal{L}_2^0}$$

$$+ \int_Z \| (t-s)^{q-1}T_V(t-s)\mathfrak{h}(s, x_s, \eta)\lambda(d\eta) \|]ds < +\infty) = 1$$

such that the following stochastic integral equation is satisfied.

$$x(t) = S_V(t)V[x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s, x_s)d\omega_2(s)] - S_V(t)g(0, x_0) + V^{-1}g(t, x_t)$$

$$+ \int_0^t (t-s)^{q-1}T_V(t-s)f(s, x_s)ds + \int_0^t (t-s)^{q-1}T_V(t-s)\sigma_1(s, x_s)d\omega_1(s)$$

$$+ \int_0^t \int_Z (t-s)^{q-1}T_V(t-s)\mathfrak{h}(s, x_s, \eta)\tilde{N}(ds, d\eta) + \sum_{0<t_k<t} T_V(t-t_k)I_k(x_{t_k}), k = 1, 2, \ldots, n. \quad (2)$$

where $S_V(t) = \int_0^\infty V^{-1}\zeta_q(\theta)S(t^q\theta)d\theta$, $T_V(t) = q\int_0^\infty V^{-1}\theta\zeta_q(\theta)S(t^q\theta)d\theta$ and $\zeta_q(\theta) = \frac{1}{\pi q}\sum_{n=1}^\infty (-\theta)^{n-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta \in [0, \infty]$ is the probability density function defined on $[0, \infty]$, which satisfies $\zeta_q(\theta) \geq 0$ and $\int_0^\infty \zeta_q(\theta)d\theta = 1$.

(iii) ${}^V D_t^{1-q}[x(t)]|_{t=0} = \sigma_2(t, x_t)\frac{d\omega_2(t)}{dt}$.

In order to prove the main result, we need the following assumptions:

(A1) The operators $V: D(V) \subset X \to Y$ and $A: D(A) \subset X \to Y$ satisfies the following properties:

(i) A and V are linear operators, and M is closed.

(ii) $D(V) \subset D(A)$ and V is bijective.

(iii) $V^{-1}: Y \to D(V) \subset X$ is a linear compact operator.

From (iii) of assumption (A1), we deduce that V^{-1} is bounded operators. Note (iii) also implies that V is closed, since the fact V^{-1} is closed and injective, then its inverse is also closed. From (i)- (iii) of assumption (A1) and the closed graph theorem, we get the boundedness of the linear operator $AV^{-1}: Y \to Y$. Consequently, AV^{-1} generates a semigroup$\{S(t) := e^{\mathcal{A}V^{-1}t}, t \geq 0\}$. Assume that $\mathcal{A} := \sup_{t\geq 0} \| S(t) \| < \infty$.

Remark 1: [22] According to the definition of V and A, it is suitable to rewrite problem (1) as the equivalent integral equation

$$Vx(t) = Vx(0) - g(0, x_0) + V^{-1}g(t, x_t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}[Ax(s) + f(s, x_s)]ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\sigma_1(s, x_s)d\omega_1(s)$$

$$+ \frac{1}{\Gamma(q)}\int_0^t \int_Z (t-s)^{q-1}\mathfrak{h}(s, x_s, \eta)\tilde{N}(ds, d\eta) + \sum_{0<t_k<t} I_k(x_{t_k}), k = 1, 2, \ldots, n.$$

and it exist.

We note that

(i) For the nonlocal condition, the function $x(0)$ is dependent on t.

(ii) The Riemann Liouville fractional derivative of $x(0)$ is well defined and ${}^{V}D_t^{1-q}x(0) \neq 0$.

(iii) The function x(0) takes the form $x_0 + \frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s,x_s)d\omega_2(s)$, where $x(0)|_{t=0} = x_0$.

(iv) The explicit and implicit integrals given in the above integral equation exist (taken in Bochners sense).

(A2) The linear and bounded operators $S_V(t)$ and $T_V(t)$ are compact for $t \geq 0$, and for any $x \in X$

$$\| S_V(t)x \| \leq \mathcal{M} \| V^{-1} \| \| x \|,$$

$$\| T_V(t)x \| \leq \mathcal{M}\frac{\| V^{-1} \|}{\Gamma(q)} \| x \|.$$

(A3) The functions $f: J \times X \to Y$, $g: J \times X \to Y$, $\sigma_1: J \times X \to \mathcal{L}_2^0$ and $\sigma_2: J \times X \to \mathcal{L}_2^0$ are continuous and measurable functions. Then there exists a function $\mathfrak{N}(t,u): J \times \mathbb{R}_+ \to \mathbb{R}_+$, $(t,u) \to \mathfrak{N}(t,u)$ and $x_s(s) \in \mathcal{L}^{\mathrm{p}}(\Omega, \mathfrak{F}_T, H_2)$ for any fixed $s \geq 0$ such that

$$\int_0^t \mathbb{E} \| f(s,x_s) \|^{\mathrm{p}} ds + \int_0^t \mathbb{E} \| g(s,x_s) \|^{\mathrm{p}} ds + \int_0^t \mathbb{E} \| \sigma_1(s,x_s) \|_{\mathcal{L}_2^0}^{\mathrm{p}} ds + \int_0^t \mathbb{E} \| \sigma_2(s,x_s) \|_{\mathcal{L}_2^0}^{\mathrm{p}} ds \leq \int_0^t \mathfrak{N}(s, \mathbb{E}(\| x_s \|)^{\mathrm{p}})ds.$$

(A4) $\mathfrak{N}(t,u)$ is locally integrable in t for each fixed $u \in \mathbb{R}_+$ and $\mathfrak{N}(t,u)$ is continuous non-decreasing in u for each $t \in J$ and for all $K > 0, u_0 \geq 0$ the integral equation

$$u(t) = u_0 + K\int_0^t \mathfrak{N}(s,u(s))ds$$

has a global solution on J.

(A5) The function I_k ia continuous and there exists constant $d_k, k = 1,2,\ldots,n$ such that for every $x \in X$ and $k = 1,2,\ldots,n$, we have

$$\mathbb{E} \| I_k(x) \|^{\mathrm{p}} \leq d_k, \text{where } d = \sum_{k=1}^{n} d_k.$$

(A6) There exists a function $\mathcal{G}(t,u): J \times \mathbb{R}_+ \to \mathbb{R}_+$, $(t,u) \to \mathcal{G}(t,u)$ and $x_s(s), x_s(s) \in \mathcal{L}^{\mathrm{p}}(\Omega, \mathfrak{F}_T, H_2)$ for any fixed $s \geq 0$ such that

$$\int_0^t \mathbb{E} \| f(t,x_s) - f(t,x_s) \|^{\mathrm{p}} ds + \int_0^t \mathbb{E} \| g(s,x_s) - g(s,x_s) \|^{\mathrm{p}} ds$$

$$+ \int_0^t \mathbb{E} \| \sigma_1(t,x_s) - \sigma_1(t,x_s) \|_{\mathcal{L}_2^0}^{p} ds + \int_0^t \mathbb{E} \| \sigma_2(t,x_s) - \sigma_2(t,x_s) \|_{\mathcal{L}_2^0}^{\mathrm{p}} ds \leq \int_0^t \mathcal{G}(s, \mathbb{E}(\| x_s - x_s \|)^{\mathrm{p}})ds,$$

and if a non-negative continuous function $z(t), t \in J$ satisfies

$$\begin{cases} z(t) \leq C\int_0^t \mathcal{G}(s,z(s))ds, t \in J \\ z(0) = 0 \end{cases}$$

where $C > 0$ is a constant, then $z(t) = 0$ for all $t \in J$.

Remark 2: Let $\mathcal{G}(t,u) = \eta(t)\vartheta(t), t \geq 0, u \in \mathbb{R}^+$ where $\eta(t) \geq 0$ is locally integrable and $\vartheta: \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, monotone non-decreasing and concave function with $\vartheta(0) = 0, \vartheta(u) > 0$ for $u > 0$ and it holds that $\int_{0+} \frac{1}{\vartheta(u)} du = \infty$. Then it is clear that ϑ satisfies assumption A4 [39].

Let us give some concrete functions $\vartheta(.)$ and let $\in (0,1)$ be sufficiently small. Define

$$\vartheta_1(u) = \begin{cases} u\log(u^{-1}), 0 \leq u \leq \epsilon \\ \epsilon\log(\epsilon^{-1}) + \vartheta_1'(\epsilon_-)(u-\epsilon), u > \epsilon \end{cases}$$

$$\vartheta_2(u) = \begin{cases} u\log(u^{-1})\log\log(u^{-1}), 0 \leq u \leq \epsilon \\ \epsilon\log(\epsilon^{-1})\log\log(\epsilon^{-1}) + \vartheta_2'(\epsilon_-)(u-\epsilon), u > \epsilon, \end{cases}$$

where ϑ_1' and ϑ_2' are the left derivative of ϑ_1 and ϑ_2 at the point ϵ. All the functions are concave, nondecreasing and satisfy $\int_{0+} \frac{1}{\vartheta_1(u)} du = +\infty$. For other examples of the function $\vartheta(.)$ refer ([24, 27, 28] and [40]).

Next we shall prove the existence and uniqueness of mild solution for the equation (1).

Theorem 3.4 Assume that the assumptions $(A1)$-$(A6)$ hold. Then the fractional system (1) has a solution in $\mathcal{B}_T$.

Proof. We shall prove the existence part of this theorem based on the Picard type approximate technique. Let us construct the sequence of stochastic process $\{x^n\}$ defined as follows.

$$(x^0)(t) = S_V(t)Vx_0$$

$$(x^{n+1})(t) = S_V(t)Vx_0 + S_V(t)V\frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s,x_s)d\omega_2(s)$$

$$-S_V(t)g(0,x_0) + V^{-1}g(t,x_t) + \int_0^t (t-s)^{q-1}T_V(t-s)f(s,x_s)ds + \int_0^t (t-s)^{q-1}T_V(t-s)\sigma_1(s,x_s)d\omega_1(s)$$

$$+\int_0^t \int_Z (t-s)^{q-1}T_V(t-s)\mathfrak{h}(s,x_s,\eta)\widetilde{N}(ds,d\eta)+\sum_{0<t_k<t} T_V(t-t_k)I_k(x_{t_k}), k=1,2,\ldots,n. \tag{3}$$

In order to prove this theorem, we need to prove the following two Lemmas. □

Lemma 3.5 Assume that the assumptions $(A1)$-$(A5)$ hold. Then $\{x^n(t)\}_{n\geq 0}$ is well defined and is bounded in $\mathcal{B}_T$, i.e., $\sup_{n\geq 0}\mathbb{E}\parallel x^n\parallel_{\mathcal{B}_T}\leq C,$ where C is a constant.

Proof. Now we have

$$\mathbb{E}\parallel(x^{n+1})(t)\parallel^{\mathfrak{p}}\leq 8^{\mathfrak{p}-1}\mathbb{E}\parallel S_V(t)Vx_0\parallel^{\mathfrak{p}}+8^{\mathfrak{p}-1}\mathbb{E}\parallel S_V(t)V\frac{1}{\Gamma(1-q)}\int_0^t(t-s)^{-q}\sigma_2(s,x_s)d\omega_2(t)\parallel^{\mathfrak{p}}$$

$$+8^{\mathfrak{p}-1}\mathbb{E}\parallel S_V(t)g(0,x_0)\parallel^{\mathfrak{p}}+8^{p-1}\mathbb{E}\parallel V^{-1}g(t,x_t)\parallel^{\mathfrak{p}}+8^{\mathfrak{p}-1}\mathbb{E}\parallel\int_0^t(t-s)^{q-1}T_V(t-s)f(s,x_s)ds\parallel^{\mathfrak{p}}$$

$$+8^{\mathfrak{p}-1}\mathbb{E}\parallel\int_0^t(t-s)^{q-1}T_V(t-s)\sigma_1(s,x_s)d\omega_1(s)\parallel^{\mathfrak{p}}+8^{\mathfrak{p}-1}\mathbb{E}\parallel\int_0^t\int_Z(t-s)^{q-1}T_V(t-s)\mathfrak{h}(s,x_s,\eta)\widetilde{N}(ds,d\eta)\parallel^{\mathfrak{p}}$$

$$+8^{\mathfrak{p}-1}\mathbb{E}\parallel\sum_{0<t_k<t}T_V(t-t_k)I_k\big(x_{t_k}\big)\parallel^{\mathfrak{p}}, k=1,2,\ldots,n.$$

If the right side terms are represented by $I_i, i=1,2,\ldots 8,$ then $\mathbb{E}\parallel(x^{n+1})(t)\parallel^{\mathfrak{p}}\leq\sum_{i=1}^8 I_i$. Evaluating each term separately, we get

$$I_1\leq 8^{\mathfrak{p}-1}\mathbb{E}\parallel S_V(t)Vx_0\parallel^{\mathfrak{p}}\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\parallel V^{-1}\parallel^{\mathfrak{p}}\parallel V\parallel^{\mathfrak{p}}\mathbb{E}\parallel x_0\parallel^{\mathfrak{p}}\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\mathbb{E}\parallel x_0\parallel^{\mathfrak{p}}\leq K_1, \tag{4}$$

where $K_1=8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\mathbb{E}\parallel x_0\parallel^{\mathfrak{p}}$.

Applying Holder inequality, Lemma 3.2, Assumptions (A2)-(A3), and using monotonicity of $\mathfrak{N}$, we get

$$I_2\leq 8^{\mathfrak{p}-1}\mathbb{E}\parallel S_V(t)V\frac{1}{\Gamma(1-q)}\int_0^t(t-s)^{-q}\sigma_2(s,x_s)d\omega_2(t)\parallel^{\mathfrak{p}}$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\parallel V^{-1}\parallel^{\mathfrak{p}}\parallel V\parallel^{\mathfrak{p}}\frac{1}{\Gamma(1-q)^{\mathfrak{p}}}\text{x } C_{\mathfrak{p}}\Big(\int_0^t(t-s)^{-2q\frac{\mathfrak{p}}{\mathfrak{p}-2}}ds\Big)^{\frac{\mathfrak{p}-2}{2}}\Big(\int_0^t\mathbb{E}\parallel(s,x_s^n(s))\parallel_{\mathcal{L}_2^0}^{\mathfrak{p}}ds\Big)$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\frac{1}{\Gamma(1-q)^{\mathfrak{p}}}C_{\mathfrak{p}}\Big(\frac{T^{\frac{-2\mathfrak{p}q}{\mathfrak{p}-2}+1}}{\frac{-2\mathfrak{p}q}{\mathfrak{p}-2}+1}\Big)^{\frac{\mathfrak{p}-2}{2}}\int_0^t\mathfrak{N}(s,\mathbb{E}\parallel x_s^n\parallel_{\mathcal{B}_s}^{\mathfrak{p}})ds$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\frac{1}{\Gamma(1-q)^{\mathfrak{p}}}C_{\mathfrak{p}}\Big(\frac{\mathfrak{p}-2}{\mathfrak{p}-2\mathfrak{p}q-2}\Big)^{\frac{\mathfrak{p}-2}{2}}T^{\frac{\mathfrak{p}-2\mathfrak{p}q-2}{2}}\int_0^t\mathfrak{N}(s,\mathbb{E}\parallel x_s^n\parallel_{\mathcal{B}_s}^{\mathfrak{p}})ds$$

$$\leq K_2\int_0^t\mathfrak{N}(s,\mathbb{E}\parallel x_s^n\parallel_{\mathcal{B}_s}^{\mathfrak{p}})ds, \tag{5}$$

where $K_2=8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\frac{1}{\Gamma(1-q)^{\mathfrak{p}}}C_{\mathfrak{p}}\big(\frac{\mathfrak{p}-2}{\mathfrak{p}-2\mathfrak{p}q-2}\big)^{\frac{\mathfrak{p}-2}{2}}T^{\frac{\mathfrak{p}-2\mathfrak{p}q-2}{2}}$.

$$I_3\leq 8^{\mathfrak{p}-1}\mathbb{E}\parallel S_V(t)g(0,x_t/_{t=0})\parallel^{\mathfrak{p}}$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\parallel V^{-1}\parallel^{\mathfrak{p}}\mathbb{E}\parallel g(0,x_0)\parallel^{\mathfrak{p}}$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\parallel V^{-1}\parallel^{\mathfrak{p}}C_1$$

$$\leq K_3, \tag{6}$$

where $K_3=8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\parallel V^{-1}\parallel^{\mathfrak{p}}C_1$.

$$I_4\leq 8^{p-1}\mathbb{E}\parallel V^{-1}g(t,x_t)\parallel^{\mathfrak{p}}$$

$$\leq 8^{p-1}\parallel V^{-1}\parallel^{\mathfrak{p}}\mathbb{E}\parallel g(t,x_t)\parallel^{\mathfrak{p}}$$

$$\leq 8^{p-1}\parallel V^{-1}\parallel^{\mathfrak{p}}C_2$$

$$\leq K_4, \tag{7}$$

here $K_4 = 8^{\mathfrak{p}-1} \parallel V^{-1} \parallel^{\mathfrak{p}} C_2$.

By the Holder inequality, Assumptions (A2)-(A3), and using monotonicity of $\mathfrak{N}$, we have

$$I_5 \leq 8^{\mathfrak{p}-1}\mathbb{E} \parallel \int_0^t (t-s)^{q-1} T_V(t-s) f(s, x_s) ds \parallel^{\mathfrak{p}}$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}} \frac{\parallel V^{-1} \parallel^{\mathfrak{p}}}{\Gamma(q)^{\mathfrak{p}}} C_{\mathfrak{p}}(\int_0^t (t-s)^{\frac{q-1}{\mathfrak{p}-1}} ds)^{\mathfrak{p}(\mathfrak{p}-1)}(\int_0^t \mathbb{E} \parallel f(s, x_s^n(s)) \parallel^{\mathfrak{p}} ds)$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}} \frac{\parallel V^{-1} \parallel^{\mathfrak{p}}}{\Gamma(q)^{\mathfrak{p}}} C_{\mathfrak{p}}(\frac{T^{\frac{pq-p}{p-1}+1}}{\frac{pq-p}{p-1}+1})^{\mathfrak{p}-1} \int_0^t \mathfrak{N}(s, \mathbb{E} \parallel x_s^n \parallel_{\mathcal{B}_s}^{\mathfrak{p}}) ds$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}} \frac{\parallel V^{-1} \parallel^{\mathfrak{p}}}{\Gamma(q)^{\mathfrak{p}}} C_{\mathfrak{p}}(\frac{\mathfrak{p}-1}{\mathfrak{p}q-1})^{\mathfrak{p}-1} T^{\mathfrak{p}q-1} \int_0^t \mathfrak{N}(s, \mathbb{E} \parallel x_s^n \parallel_{\mathcal{B}_s}^{\mathfrak{p}}) ds$$

$$\leq K_5 \int_0^t \mathfrak{N}(s, \mathbb{E} \parallel x_s^n \parallel_{\mathcal{B}_s}^{\mathfrak{p}}) ds, \tag{8}$$

here $K_5 = 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}} \frac{\parallel V^{-1} \parallel^{\mathfrak{p}}}{\Gamma(q)^{\mathfrak{p}}} C_{\mathfrak{p}}(\frac{\mathfrak{p}-1}{\mathfrak{p}q-1})^{\mathfrak{p}-1} T^{\mathfrak{p}q-1}$.

By the Holder inequality, Lemma 3.2, Assumptions (A2)-(A3), and using monotonicity of $\mathfrak{N}$, we have

$$I_6 \leq 8^{\mathfrak{p}-1}\mathbb{E} \parallel \int_0^t (t-s)^{q-1} T_V(t-s) \sigma_1(s, x_s) d\omega_1(t) \parallel^{\mathfrak{p}}$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}} \frac{\parallel V^{-1} \parallel^{\mathfrak{p}}}{\Gamma(q)^{\mathfrak{p}}} C_{\mathfrak{p}}(\int_0^t (t-s)^{2(q-1)\frac{\mathfrak{p}}{\mathfrak{p}-2}} ds)^{\frac{\mathfrak{p}-2}{2}}(\int_0^t \mathbb{E} \parallel \sigma_1(s, x_s^n(s)) \parallel_{\mathcal{L}_2^0}^{\mathfrak{p}} ds)$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}} \frac{\parallel V^{-1} \parallel^{\mathfrak{p}}}{\Gamma(q)^{\mathfrak{p}}} C_{\mathfrak{p}}(\frac{T^{\frac{2\mathfrak{p}(q-1)}{\mathfrak{p}-2}+1}}{\frac{2\mathfrak{p}(q-1)}{\mathfrak{p}-2}+1})^{\frac{\mathfrak{p}-2}{2}} \int_0^t \mathfrak{N}(s, \mathbb{E} \parallel x_s^n \parallel_{\mathcal{B}_s}^{\mathfrak{p}}) ds$$

$$\leq 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}} \frac{\parallel V^{-1} \parallel^{\mathfrak{p}}}{\Gamma(q)^{\mathfrak{p}}} C_{\mathfrak{p}}(\frac{\mathfrak{p}-2}{2\mathfrak{p}q-\mathfrak{p}-2})^{\frac{\mathfrak{p}-2}{2}} T^{\frac{2\mathfrak{p}q-\mathfrak{p}-2}{2}} \int_0^t \mathfrak{N}(s, \mathbb{E} \parallel x_s^n \parallel_{\mathcal{B}_s}^{\mathfrak{p}}) ds$$

$$\leq K_6 \int_0^t \mathfrak{N}(s, \mathbb{E} \parallel x_s^n \parallel_{\mathcal{B}_s}^{\mathfrak{p}}) ds, \tag{9}$$

here $K_6 = 8^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}} \frac{\parallel V^{-1} \parallel^{\mathfrak{p}}}{\Gamma(q)^{\mathfrak{p}}} C_{\mathfrak{p}}(\frac{\mathfrak{p}-2}{2\mathfrak{p}q-\mathfrak{p}-2})^{\frac{\mathfrak{p}-2}{2}} T^{\frac{2\mathfrak{p}q-\mathfrak{p}-2}{2}}$.

By using Holder inequality, Lemma 2.4, Lemma 3.2 and Assumption (A2), we have

$$I_7 \leq 8^{\mathfrak{p}-1}\mathbb{E} \parallel \int_0^t \int_Z (t-s)^{q-1} T_V(t-s) \mathfrak{h}(s, x_s, \eta) \widetilde{N}(ds, d\eta) \parallel_X^{\mathfrak{p}}$$

$$\leq 8^{\mathfrak{p}-1}\mathbb{E}(\parallel \int_0^t \int_Z (t-s)^{q-1} T_V(t-s) \mathfrak{h}(s, x_s, \eta) \widetilde{N}(ds, d\eta) \parallel_X^2)^{\frac{\mathfrak{p}}{2}}$$

$$\leq 8^{\mathfrak{p}-1} C_{\mathfrak{p}}(\int_0^t (t-s)^{2(q-1)\frac{\mathfrak{p}}{\mathfrak{p}-2}} ds)^{\frac{\mathfrak{p}-2}{2}} \mathbb{E}(\parallel \int_0^t \int_Z T_V(t-s) \mathfrak{h}(s, x_s, \eta) \lambda(d\eta ds) \parallel_X^2)^{\frac{\mathfrak{p}}{2}}$$

$$\leq 8^{\mathfrak{p}-1} C_{\mathfrak{p}}(\frac{T^{\frac{2\mathfrak{p}(q-1)}{\mathfrak{p}-2}+1}}{\frac{2\mathfrak{p}(q-1)}{\mathfrak{p}-2}+1})^{\frac{\mathfrak{p}-2}{2}} \mathbb{E}(\parallel \int_0^t \int_Z T_V(t-s) \mathfrak{h}(s, x_s, \eta) \lambda(d\eta ds) \parallel_X^2)^{\frac{\mathfrak{p}}{2}}$$

$$\leq 8^{\mathfrak{p}-1} C_{\mathfrak{p}}(\frac{\mathfrak{p}-2}{2\mathfrak{p}q-\mathfrak{p}-2})^{\frac{\mathfrak{p}-2}{2}} T^{\frac{2\mathfrak{p}q-\mathfrak{p}-2}{2}} \mathbb{E}(\sup_{0 \leq \tau \leq t} \parallel \int_0^t \int_Z T_V(\tau-s) \mathfrak{h}(s, x_s, \eta) \lambda(d\eta ds) \parallel_X^2)^{\frac{\mathfrak{p}}{2}}$$

$$\leq 8^{\mathfrak{p}-1} C_{\mathfrak{p}_1} \mathcal{M}^{\mathfrak{p}} \frac{\parallel V^{-1} \parallel^{\mathfrak{p}}}{\Gamma(q)^{\mathfrak{p}}} [\int_0^t \int_Z \mathbb{E} \parallel \mathfrak{h}(s, x_s, \eta) \parallel_X^2 \lambda(d\eta) ds$$

$$+(\int_0^t \int_Z \mathbb{E} \parallel \mathfrak{h}(s,x_s,\eta) \parallel_X^4 \lambda(d\eta)ds)^{\frac{1}{2}}]^{\frac{p}{2}}$$

$$\leq 8^{p-1}\mathcal{M}^p \frac{\parallel V^{-1} \parallel^p}{\Gamma(q)^p} C_{p_1} C_3 < \infty$$

$$\leq K_7 < \infty, \tag{10}$$

here $K_7 = 8^{p-1}\mathcal{M}^p \frac{\parallel V^{-1}\parallel^p}{\Gamma(q)^p} C_{p_1} C_3$.

$$I_8 \leq 8^{p-1}\mathbb{E} \parallel \sum_{0<t_k<t} T_V(t-t_k) I_k(x_{t_k}) \parallel^p, k = 1,2,\ldots,n$$

$$\leq 8^{p-1}\mathcal{M}^p \frac{\parallel V^{-1} \parallel^p}{\Gamma(q)^p} \sum_{k=1}^n d_k$$

$$\leq 8^{p-1}\mathcal{M}^p \frac{\parallel V^{-1} \parallel^p}{\Gamma(q)^p} d$$

$$\leq K_8, \tag{11}$$

here $K_8 = 8^{p-1}\mathcal{M}^p \frac{\parallel V^{-1}\parallel^p}{\Gamma(q)^p} d$.

Hence we have

$$\mathbb{E} \parallel (x^{n+1})(t) \parallel_{\mathcal{B}_t}^p \leq K_1 + K_3 + K_4 + K_7 + K_8 + (K_2 + K_5 + K_6) \int_0^t \mathfrak{N}(s, \mathbb{E} \parallel x_s^n \parallel_{\mathcal{B}_s}^p) ds, \leq K_9 + K_{10} \int_0^t \mathfrak{N}(s, \mathbb{E} \parallel x_s^n \parallel_{\mathcal{B}_s}^p) ds, \tag{12}$$

where $K_9 = K_1 + K_3 + K_4 + K_7 + K_8$ and $K_{10} = K_2 + K_5 + K_6$.

Comparing with the integral equation

$$z(t) = K_9 + K_{10} \int_0^t \mathfrak{N}(s, z(s)) ds \tag{13}$$

and using (A4), we find that equation (13) has a global solution $z(\cdot)$ on J.

Next we shall show that $\mathbb{E} \parallel x^n \parallel_{\mathcal{B}_t}^p \leq z(t)$ for all $t \in J, n \geq 0$, by using the method of induction.

$$\mathbb{E} \parallel x^n \parallel_{\mathcal{B}_t}^p = \sup_{0\leq s\leq t} \mathbb{E} \parallel S_V(t) V x_0 \parallel^p$$

$$\leq \mathcal{M}^p \parallel V^{-1} \parallel^p \parallel V \parallel^p \mathbb{E} \parallel x_0 \parallel^p$$

$$\leq \mathcal{M}^p \mathbb{E} \parallel x_0 \parallel^p$$

$$\leq K_{11} \leq z(t)\ \forall t \in J.$$

Let us assume that $\mathbb{E} \parallel x^n \parallel_{\mathcal{B}_t}^p \leq z(t)$. Then by (12), the method of mathematical induction and the non-decreasing property of $\mathfrak{N}$ in the second variable, we get

$$z(t) - \mathbb{E} \parallel (x^{n+1})(t) \parallel_{\mathcal{B}_t}^p \geq K_{10} \int_0^t (\mathfrak{N}(s, z(s)) - \mathfrak{N}(s, \mathbb{E} \parallel x_s^n \parallel_{\mathcal{B}_s}^p)) ds \geq 0, \forall t \in J.$$

In particular $\mathbb{E} \parallel x^n \parallel_{\mathcal{B}_T} \leq z(T)^{\frac{1}{p}}$ and hence $\{x^n\}_{n\geq 0}$ is well-defined. □

Lemma 3.6 *Let assumptions (A1)-(A6) hold. Then the sequence* $\{x^n\}_{n\geq 0}$ *is a Cauchy sequence in the space* $\mathcal{B}_T$.

Proof. Let

$$\delta_n(t) = \sup_{m\geq n} \mathbb{E} \parallel x^m - x^n \parallel_{\mathcal{B}_t}^p$$

By applying the same argument as in the Lemma 3.5, we get

$$\mathbb{E} \parallel x^m - x^n \parallel_{\mathcal{B}_t}^p \leq C \int_0^t \mathcal{G}(s, \mathbb{E} \parallel x^{m-1} - x^{n-1} \parallel_{\mathcal{B}_s}^p) ds, \forall t \in J,$$

Where

$$C = 3^{\mathfrak{p}-1}\mathcal{M}^{\mathfrak{p}}\frac{\| V^{-1} \|^{\mathfrak{p}}}{\Gamma(q)^{\mathfrak{p}}}C_{\mathfrak{p}}(\frac{\Gamma(q)^{2\mathfrak{p}}\sin^{\mathfrak{p}}\pi q}{\pi^{\mathfrak{p}} \| V^{-1} \|^{\mathfrak{p}}}(\frac{\mathfrak{p}-2}{\mathfrak{p}-2\mathfrak{p}q-2})^{\frac{\mathfrak{p}-2}{2}}T^{\frac{\mathfrak{p}-2\mathfrak{p}q-2}{2}} + (\frac{\mathfrak{p}-1}{\mathfrak{p}q-1})^{\mathfrak{p}-1}T^{\mathfrak{p}q-1} + (\frac{\mathfrak{p}-2}{2\mathfrak{p}q-\mathfrak{p}-2})^{\frac{\mathfrak{p}-2}{2}}T^{\frac{2\mathfrak{p}q-\mathfrak{p}-2}{2}}).$$

Which implies that

$$\delta_n(t) \leq C\int_0^t \mathcal{G}(s,\delta_{n-1}(s))ds.$$

By the Lemma 3.5, it is clear that the functions $\delta_n, n \geq 0$, are well-defined, uniformly bounded and monotone non-decreasing. Since$\{\delta_n\}_{n\geq 0}$ is a monotone non-increasing sequence for each $t \in J$, there exists a monotone non-decreasing function δ such that $\lim_{n\to+\infty}\delta_n(t) = \delta(t)$

Taking limit as $n \to +\infty$ in the above inequality and by using the Lebesgue convergence theorem, we get

$$\delta(t) \leq K_{12}\int_0^t \mathcal{G}(s,\delta(s))ds.$$

From the assumption (A6) and [41], we have $\delta = 0$, for all $t \in J$. But $0 \leq \mathbb{E} \| x^m - x^n \|_{\mathcal{B}_T}^{\mathfrak{p}} \leq \delta_n(T)$ and $\delta_n(T) \to \delta(T) = 0$ as $n \to +\infty$. Hence the sequence $\{x^n\}_{n\geq 0}$ is a Cauchy sequence in the space $\mathcal{B}_T$. □

Proof of the main Theorem 3.2.

(1) Existence of solution: Let x be the limit of the sequence $\{x^n\}_{n\geq 0}$. Then from Lemma 3.6, it is clear that the right side of the second equality of (3) tends to

$$S_V(t)Vx_0 + S_V(t)V\frac{1}{\Gamma(1-q)}\int_0^t (t-s)^{-q}\sigma_2(s,x_s)d\omega_2(s)$$

$$-S_V(t)g(0,x_0) + V^{-1}g(t,x_t) + \int_0^t (t-s)^{q-1}T_V(t-s)f(s,x_s)ds + \int_0^t (t-s)^{q-1}T_V(t-s)\sigma_1(s,x_s)d\omega_1(s)$$

$$+\int_0^t\int_Z (t-s)^{q-1}T_V(t-s)\mathfrak{h}(s,x_s,\eta)\widetilde{N}(ds,d\eta) + \sum_{0<t_k<t} T_V(t-t_k)I_k(x_{t_k}), k = 1,2,\ldots,n \text{ as } n \to +\infty.$$

(2) Uniqueness of solution: Let $x_1 \in \mathcal{B}_T$ and $x_2 \in \mathcal{B}_T$ be two solutions of equation (1). Using the similar argument as in Lemma 3.6, we have

$$\mathbb{E} \| x_1 - x_2 \|_{\mathcal{B}_t}^{\mathfrak{p}} \leq C\int_0^t \mathcal{G}(s,\mathbb{E} \| x_1 - x_2 \|_{\mathcal{B}_s}^{\mathfrak{p}})ds, \forall t \in J,$$

Using (A6), we get $\mathbb{E} \| x_1 - x_2 \|_{\mathcal{B}_t}^{\mathfrak{p}} = 0$ for all $t \in J$ and hence $\mathbb{E} \| x_1 - x_2 \|_{\mathcal{B}_T}^{\mathfrak{p}} = 0$. So which gves that $x_1 = x_2$. □

4. Application

Consider the following fractional partial stochastic nonlocal differential system of Sobolev type

$${}^cD_t^q[z(t,x) - z_{xx}(t,x) - g(t,z(t-\tau,x))] = \frac{\partial^2}{\partial x^2}z(t,x) + f(t,z(t,x),z(t-\tau,x))dt + \sigma_1(t,z(t,x),z(t-\tau,x))\frac{d\omega_1(t)}{dt}$$

$$+\int_Z \eta(\mathfrak{h}(t,z(t,x),z(t-\tau,x)))\widetilde{N}(dt,d\eta), (t,x) \in J\times[0,1],\ 0 \leq t \leq b, 0 \leq x \leq 1, z(t,0) = z(t,1) = 0, t \in J,$$

$$z(0,x) = z_0(x) + \frac{1}{\Gamma(1-q)}\sum_{k=1}^m b_k\int_0^t (t-s)^{-q}z(t_k-\tau,x)d\omega_2(s), x \in [0,1], t \in J,$$

$$[z(t_k^+) - z(t_k^-)]x = I_k(z(t_k))x = \int_{-\infty}^t c_k(t_k-s)z(s,x)ds, k = 1,2,\ldots,n, \quad (14)$$

where ${}^cD_t^q$ is the Caputo fractional partial derivative of order $0 < q \leq 1$. c_k are continuous for $k = 1,2,..,n$ and b_k, $k = 1,2,\ldots,m$ are fixed numbers. Define the functions

$$z(t)x = z(t,x), f(t,x_t)(z) = f\big(t,z(t,x),z(t-\tau,x)\big), g(t,x_t)(z) = g\big(t,z(t,x),z(t-\tau,x)\big),$$

$$\sigma_1(t,x_t)(z) = \sigma_1(t,z(t,x),z(t-\tau,x))$$

and $\sigma_2(t,x_t)(z) = \sum_{k=1}^m b_kz(t_k-\tau,x)$. $I_k: X \to Y$ is an appropriate function. Let $0 = t_0 < t_1 < t_2 <\ldots< t_n < b$ be the given time points and the symbol $\Delta\zeta(t)$ represents the jump of the function ζ at t defined by $\Delta\zeta(t) = \zeta(t^+) - \zeta(t^-)$. Also $\omega_1(t)$ and $\omega_2(t)$ are two sided and standard one-dimensional Brownian motions defined on the filtered probability space $(\Omega,\mathfrak{F},\{\mathfrak{F}_t\}_{t\geq 0},\mathbb{P})$.

To write the above system (14) into the abstract form (1), let $X = \mathbb{E} = \mathcal{L}_2([0,1])$. Define the operators $V: D(V) \subset X \to Y$ and $A: D(A) \subset X \to Y$ by $V\xi = \xi - \xi''$ and $A\xi = -\xi''$ respectively, where the domains $D(V)$ and $D(A)$ are given by

$D(V) = D(A) = \{\xi \in X: \xi, \xi'$ are absolutely continuous, $\xi'' \in X$ and $\xi(0) = \xi(1) = 0\}$.

Then V and A can be written respectively as [18]

$$V\xi = \sum_{n=1}^{\infty} (1+n^2)\langle \xi, e_n\rangle e_n, \xi \in D(V),$$

$$A\xi = \sum_{n=1}^{\infty} -n^2\langle \xi, e_n\rangle e_n, \xi \in D(A),$$

where $e_n(x) = \sqrt{\frac{2}{\pi}}\sin nx, n = 1,2,\ldots$ is the orthogonal set of eigen functions of A. Also for auy $\xi \in X$, we have

$$V^{-1}\xi = \sum_{n=1}^{\infty} \frac{1}{1+n^2}\langle \xi, e_n\rangle e_n,$$

$$AV^{-1}\xi = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}\langle \xi, e_n\rangle e_n,$$

$$S(t)\xi = \sum_{n=1}^{\infty} \exp(\frac{-n^2 t}{1+n^2})\langle \xi, e_n\rangle e_n,$$

$$S_V(t)\xi = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{1+n^2}\int_0^{\infty} \theta\zeta_{\frac{3}{4}}(\theta)\exp(\frac{-n^2 t^{\frac{3}{4}}\theta}{1+n^2})d\theta\langle \xi, e_n\rangle e_n.$$

It is clear that V^{-1} is compact, bounded with $\| V^{-1} \| \le 1$ and AV^{-1} generates the above strongly continuous semigroup $S(t)$ on Y such that $\| S(t) \| \le e^{-t} \le 1$. Clearly the functions f, g and $\mathfrak{h}$ are continuous. Let $x(t)(\cdot) = x(t,\cdot)$. Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space and $\{K(t): t \in J\}$ be a Poisson point process taking values in the space $K = [0,\infty)$ with a σ-finite intensity measure $\lambda(dy)$. The Poisson counting measure $\widetilde{N}(dt, dy)$ is induced by $K(\cdot)$ and the compensating martingale measure is denoted by $\widetilde{N}(dt, d\eta) = N(dt, d\eta) - dt\lambda(d\eta)$. Assume that $\int_Z \mathbb{E} \| J(s,y) \|_H^2 \lambda(dy) < \infty$ and $\int_Z \mathbb{E} \| J(s,y) \|_H^4 \lambda(dy) < \infty$. Let $\mathcal{G}(t,u) = \eta(t)\vartheta(t), t \ge 0, u \in \mathbb{R}^+$ where $\eta(t) \ge 0$ is locally integrable and $\vartheta: \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, monotone non-decreasing and concave function with $\vartheta(0) = 0, \vartheta(u) > 0$ for $u > 0$ and it holds that

$$\int_{0+} \frac{1}{\vartheta(u)} du = \infty$$

and $f: X \to Y$ is a continuous function such that

$$\mathbb{E} \| f(h_1) - f(h_2) \|^2 \le \vartheta\mathbb{E} \| h_1 - h_2 \|^2, h_1, h_2 \in X.$$

Moreover $g: J \times X \to Y$ is continuous and measurable.

Hence with the above choices, the system (14) can be rewritten to the abstract form (1) and all the conditions of Theorem 3.4 are satisfied. Thus there exists a unique mild solution for the fractional stochastic nonlinear differential system of Sobolev type (14).

5. Conclusion

For the future work, the existence and uniqueness results could be extended to study sufficient conditions for stochastic nature of wave equations, heat equations, Burger's equations and Navier stochastic equations with infinite delay and Poisson jumps satisfying the fractional stochastic nonlocal conditions by using successive approximation approach, fractional calculus stochastic analysis techniques. Also we can extend the result to find the existence of solutions of non-non-Lipschitz Sobolev type fractional neutral impulsive stochastic differential equations satisfying the fractional stochastic nonlocal conditions with infinite delay and Poisson jumps of orders $\frac{1}{2} < q < 1$, $2 < q < 3$ and $2 < q \le 3$ in $\mathcal{L}_p$ space. We will investigate the control problem for neutral stochastic fractional differential equations with infinite delay Poisson jumps in $\mathcal{L}_p$ space. of course the fundamental solution theory for neutral nonlinear systems with infinite delay and Poisson jumps. Moreover, It should be mentioned that the fundamental solution theory is a very useful tool and can be applied to study many other aspects of asymptotic behaviors and control problems such as optimal control for semilinear fractional stochastic differential systems with infinite delay and Poisson jumps.

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