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Supra Topologies for Digital Plane

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Digital topology aims to transfer concepts from classical topology (such as connectivity of objects, properties of their boundary and their neighborhood, as well as continuity) to digital spaces (such as Z^2), which are used to model computer images. It has two main approaches graph theoretic and topological. Graph theoretic approach depends on 4 and 8 adjacencies which imply connectivity paradoxes. Our goal is to construct supra topological structure using 4 and 8 adjacencies to define connectivity to help in solving connectivity paradoxes. We construct 4 and 8 supra topologies. Properties of closure and interior operators are given. Examples and counter examples are obtained.

Introduction

Digital topology was first studied in the late 1960s by the computer image analysis researcher Azriel Rosenfeld (1931–2004) [9], whose publications on the subject played a major role in establishing and developing the field. The term "digital topology" was itself invented by Rosenfeld, who used it in a 1973 publication for the first time. A related work called the grid cell topology appeared in Alexandrov-Hopf's book Topologie I (1935) [7] can be considered as a link to classic combinatorial topology. Rosenfeld et al. proposed digital connectivity such as 4-connectivity and 8-connectivity in two dimensions as well as 6-connectivity and 26-connectivity in three dimensions. The labeling method for inferring a connected component was studied in the 1970s. T. Pavlidis (1982) [8] suggested the use of graph-theoretic algorithms such as the depth-first search method for finding connected components. V. Kovalevsky (1989) [10] extended Alexandrov-Hopf's 2D grid cell topology to three and higher dimensions. He also proposed (2008) [11] a more general axiomatic theory of locally finite topological spaces and abstract cell complexes formerly suggested by Steinitz (1908). The book of 2008 contains new definitions of topological balls and spheres independent of a metric and numerous applications to digital image analysis. In the early 1980s, digital surfaces were studied. Morgenthaler and Rosenfeld (1981) gave a mathematical definition of surfaces in three-dimensional digital space. This definition contains a total of nine types of digital surfaces. The digital manifold was studied in the 1990s. A recursive definition of the digital k-manifold was proposed intuitively by Chen and Zhang in 1993. Many applications were found in image processing and computer vision. Here, we use the concept of "supra topology" introduced by A.S. Mashhour et al [1]. In contrast to other concepts of digital topologies, the main advantage of this concept is that we can directly transfer many concepts from classical topology to digital topology. The purpose of the work is to find a link between graph theoretic approach and topological are by constructing a supratopology. This paper is organized as follows: In Section 3 we introduce the supratopology using 4 and 8 adjacencies. In Section 4 we introduce 4and 8 Supra topologies, gives examples of open set and not open sets and define connectedness based on 4-adjacency.

Preliminaries

The purpose of this article is to present basic definitions and facts used in the paper

4-Adjacency [3]

Two grid points $p, q \in Z^2$ are called 4-adjacent or proper 4-neighbors

Iff $p \neq q$ and $p \in N_4(q)$.

Where,

$$\mathrm{N}_4(\mathrm{p}) = \{(\mathrm{x},\mathrm{y}), (\mathrm{x}+1,\mathrm{y}), (\mathrm{x}-1,\mathrm{y}), (\mathrm{x},\mathrm{y}+1), (\mathrm{x},\mathrm{y}-1)\} = U_4(\mathrm{x},\mathrm{y}),$$
$$p = (\mathrm{x},\mathrm{y})$$

8-Adjacency [3]

Two grid points $p, q \in Z^2$ are called 8-adjacent or proper 8-neighbors

Iff $p \neq q$ and $p \in N_8(q)$.

Where,

$$\mathrm{N}_8(\mathrm{p}) = \mathrm{N}_4(\mathrm{p}) \mathrm{U} \{(\mathrm{x}+1,\mathrm{y}+1), (\mathrm{x}+1,\mathrm{y}-1), (\mathrm{x}-1,\mathrm{y}+1), (\mathrm{x}-1,\mathrm{y}-1)\} = U_8(\mathrm{x},\mathrm{y}).$$

Digital spaces [6]

A digital space is a pair (V, π) where V is a non-empty set and π is a binary, symmetric relation on V such that for any two elements x and y of V there is a finite sequence $(x^0, \dots \dots, x^n)$ of elements in V such that $x = x^0, y = x^n$ and $(x^j, x^{j+1}) \in \pi$ for $j = 0, 1, \dots, n-1$.

The relation π is often called an adjacency relation, and that $(x, y) \in \pi$ means that x and y are connected. The last requirement of the definition is that the space is connected under the given relation; that V is π-connected.

Supra-Topology

The concept of topological structures has been widely applied in many theoretical and application fields, but it depends on three conditions which limit its applications. The concept of supra topology is a general and reflexable structure, each topology is a supratopology.

Definition1 [1]:

A supra-topology on a set X is a system O of subsets of X, which meets the following conditions:

1. $\emptyset, X \in O$,

2. The union of every subfamily of O is a member of O.

Example:

If $X = \{a, b, c\}$ *then* $O = \{X, \emptyset, \{b\}, \{a, b\}\}$ is a supratopology.

Clearly, this a more general definition than the definition of a topology, because the axiom about the intersection of two open sets is omitted. This concept was first introduced by A.S. Mashhour et al. Here we refrain from reviewing the results of A.S. Mashhour et al. Instead, we deal with two special supra topologies, the 8- and 4-supra-topology. What makes these supra-topologies important to image processing is that the 8- and 4-connected sets in the corresponding supra-topology are exactly the 8- and 4-connected sets in the common graph-theoretical interpretation.

Results

A two dimensional digital picture is as usual a tuple (Z^2, B) where $B \subseteq Z^2$. The elements of Z^2 are called points of the digital picture, and the elements of B are called the black points of the picture, and the points in $Z^2 \setminus B$ are called the white points of the picture. Now, we define the 4- and 8-Supratopologies by their "point bases" $U_4(p)$ and $U_8(p)$, respectively, for $P = (P_1, P_2) \in Z^2$:

- *4-Supra topology*

$$\text{If } \mathrm{P} = (\mathrm{p}_1, \mathrm{p}_2) \text{ then}$$

$$\mathrm{U}_4(\mathrm{P}_1, \mathrm{P}_2) = \{(\mathrm{P}_1, \mathrm{P}_2 - 1), (\mathrm{P}_1 - 1, \mathrm{P}_2), (\mathrm{P}_1 + 1, \mathrm{P}_2), (\mathrm{P}_1, \mathrm{P}_2 + 1) = \mathrm{U}_4(\mathrm{p}).$$

Proposition 1:

The class $O_4 = \cup \{U_4(P) \mid \mathrm{P} \in Z^2\}$ is as 4-supratopology on Z^2.

Proof

1) $Z^2, \emptyset \in O_4$.

2) Let $p, q \in Z^2, U_4(p), U_4(q) \in O_4$ *then* $U_4(p) \cup U_4(q) \in O_4$.

So from (1) and (2) O_4 is a 4-supratopology.

(Z^2, O_4) is a 4-supratopological space, the members of O_4 are called 4-supra open sets .The following is an example for a 4-supra open set.

Example 1:

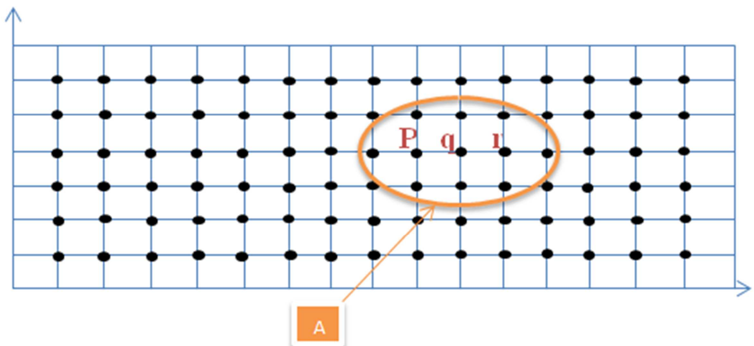

Figure 1. *4-supra open set.*

A is 4- supra open set

$$A = \{(9,3), (10,3), (11,3), (8,4), (9,4), (10,4), (11,4), (12,4), (10,5), (11,5), (12,5)\}$$

$$\mathrm{A} = \mathrm{U}_4(\mathrm{p}) \cup \mathrm{U}_4(\mathrm{q}) \cup \mathrm{U}_4(\mathrm{r})$$

The following is an example for a non 4-supra open set

Example 2:

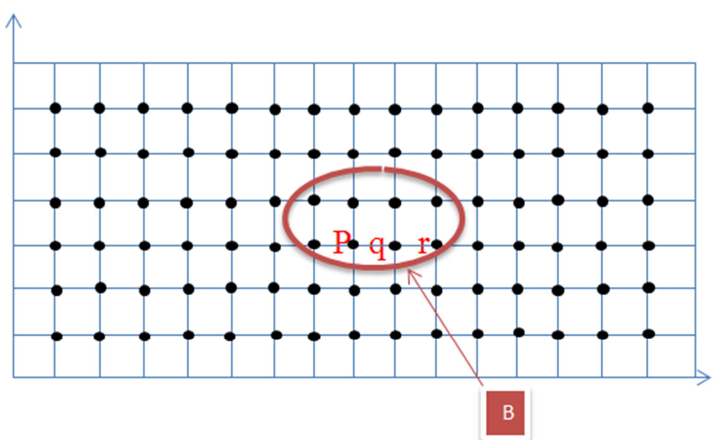

Figure 2. *Non 4-supra open set.*

B isn't a 4-supra open set

$$\text{Since } B \neq U_4(p) \cup U_4(q) \cup U_4(r)$$

- *8-Supratopology*

$$\text{If } P = (p_1, p_2) \text{then}$$

$$U_8(P_1, P_2) = \{(q_1, q_2) \in Z^2 \mid \max(|q_1 - p_1|, |q_2 - p_2| \leq 1) = U_8(p).$$

Proposition 2

The class $O_8 = \cup\, (U_8(P) \mid P \in Z^2)$ is a 8-supratopology on Z^2.

Proof

1) $Z^2, \emptyset \in O_8$.

2) Let $p, q \in Z^2, U_8(p), U_8(q) \in O_8$ *then* $U_8(p) \cup U_8(q) \in O_8$.

So from (1) and (2) O_8 is a 8-supratopology.

(Z^2, O_8) is a 8-supratopological space ,the members of O_8 are called 8-supra open sets and the following is an example for a 8-supra open set

Example 3:

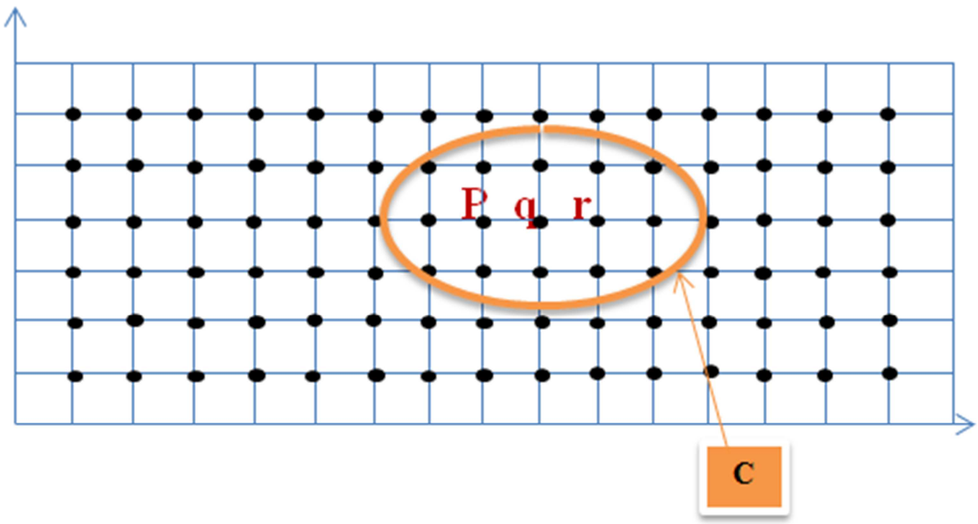

Figure 3. *8-supra open set.*

C is a 8-supra open set

$$C = \{(7,3), (8,3), (9,3), (10,3), (11,3), (7,4), (8,4), (9,4), (10,4), (11,4), (7,5)$$

$$(8,5), (9,5), (10,5), (11,5)\} = U_8(p) \cup U_8(q) \cup U_8(r)$$

The following is an example for a non 8-supra open set

Example 4:

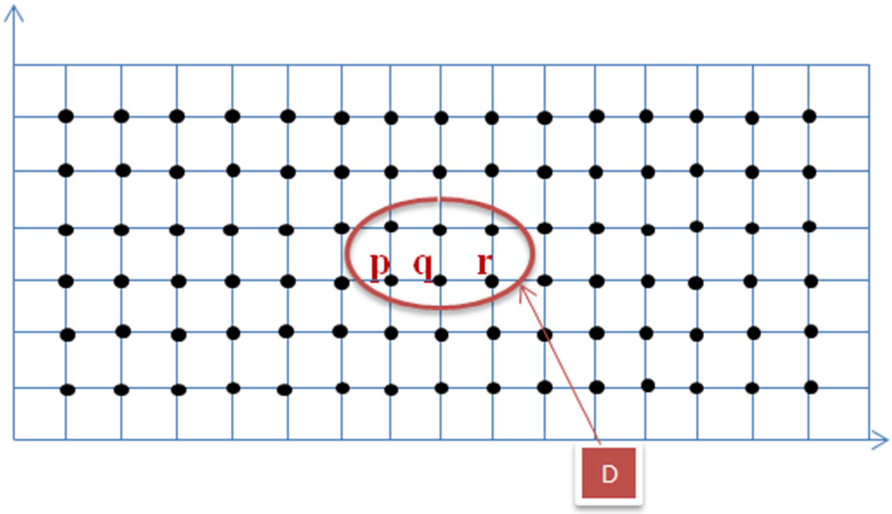

Figure 4. *Non 8-supra open set.*

● D isn't a 8-supra open set

$$\text{Since } D \neq U_8(p) \cup U_8(q) \cup U_8(r)$$

Definition 2:

If (X, τ) is a supra topological space, $B \subset X$ then among the topological properties and operators we consider the interior operator:

$$\text{Int}(B) = \{ x \in Z^2 \mid \exists\, y \in Z^2 \; with \; x \in U(y) \subseteq B\}$$

$$\text{OR Int}(B) = \cup \{U_{4or\,8}(y) : U_{4or\,8}(y) \subset B\}$$

Properties of interior operator of topology

1. $int\ (B) \subset B$.
2. $if\ A \subset B\ then\ int\ (A) \subset int(B)$.
3. $int\ (Z^2 - B) = Z^2 - cl(B)$.
4. $int\ (A) \cup int(B) \subset int\ (A \cup B)$.
5. $int\ (A) \cap int(B) = int\ (A \cap B)$.
6. $int\big(int(B)\big) = int(B)$.

Definition 3

If (X, τ) is a supra topological space, $B \subset X$ then among the topological properties and operators we consider the closure operator:

$$\text{Cl}(B) = \{ x \in Z^2 \mid \exists\, y \in Z^2 \; with \; x \in U(y) \; follows \; U(y) \cap B \neq \phi\}.$$

$$\text{OR cl}(B) = \cup \{U_{4or\,8}(y) : U_{4or\,8}(y) \cap B \neq \emptyset\}.$$

Properties of closure operator of topology

1. $B \subset cl(B)$.
2. $if\ A \subset B\ then\ cl\ (A) \subset cl(B)$.
3. $cl\ (Z^2 - B) = Z^2 - int(B)$.
4. $cl(A \cup B) = cl\ (A) \cup cl(B)$.

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5. $cl\ (A) \cap cl(B) \supset cl\ (A \cap B)$.

6. $cl\big(cl(B)\big) = cl(B)$.

The following is an example for 4-supra interior and closure

Example 5:

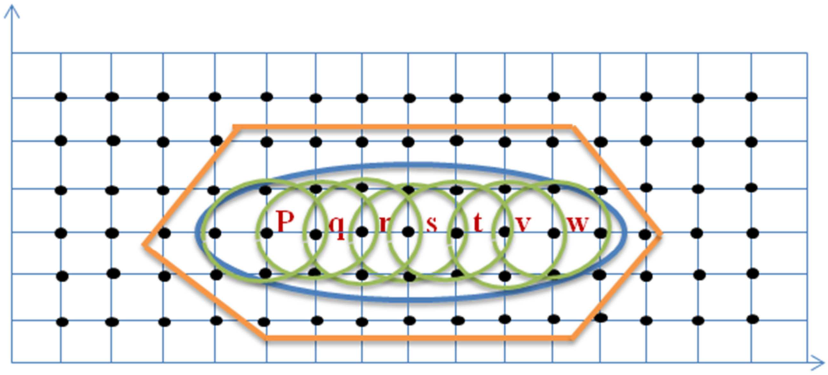

Figure 5. *4-supra interior and closure.*

Set (B)

4-Supra Interior (B) = $U_4(p) \cup U_4(q) \cup U_4(r) \cup U_4(s) \cup U_4(t) \cup U_4(v) \cup U_4(w)$

4-Supra Closure (B).

Note

Interior is equal to set because it is an open set

The following is an example for 8-supra interior and closure

Example 6:

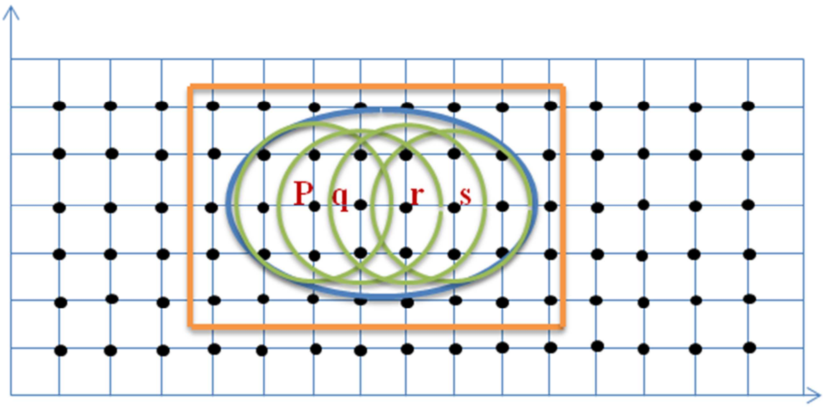

Figure 6. *8-supra interior and closure.*

Set (B)

8-Supra Interior (B) = $U_8(p) \cup U_8(q) \cup U_8(r) \cup U_8(s)$

8-Supra Closure(B)

Definition 4

If (X, τ) is a supra topological space, $B \subset X$ then Center of interior is defined as:

$$C(B^{\circ}) = \{\, x \in B : U(x) \subseteq B\}$$

The following are examples for 4 and 8 center for interior

Example 7:

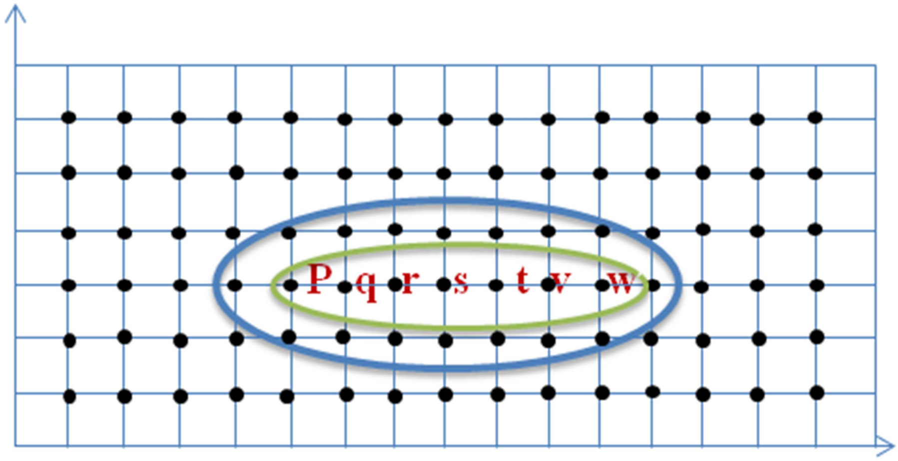

Figure 7. *4-center for interior.*

Set (B)

4-center interior (B) = p ∪ q ∪ r ∪ s ∪ t ∪ v ∪ w, since U_4 of each point is contained in B

Example 8:

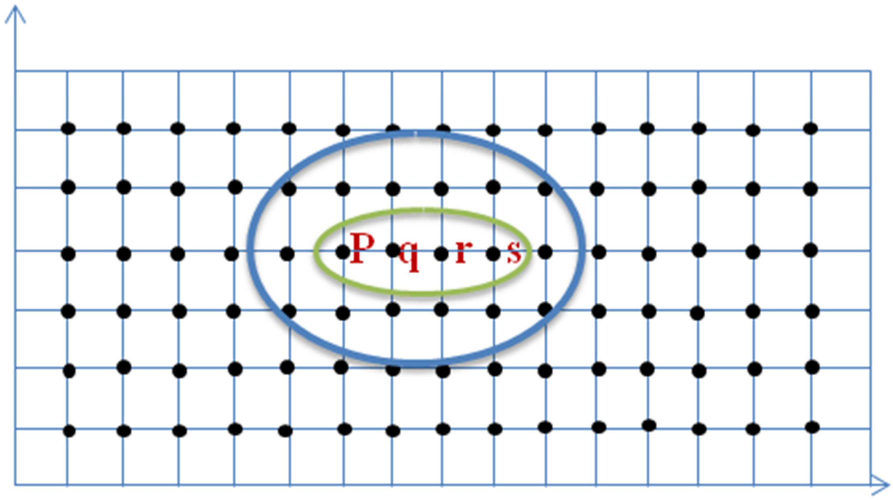

Figure 8. *8-center for interior.*

Set (B)

8-center interior (B) = p ∪ q ∪ r ∪ s, since U_8 of each point is contained in B.

Proposition 3:

Any subset B from Z^2 is a preopen and its closure is open ($\mathrm{int}\,(\overline{\mathrm{B}}) = \overline{\mathrm{B}}$).

Proof:

$$\text{Since } B \subset \bar{B} = \bar{B}^{\circ}$$

$$\therefore B \subset \bar{B}^{\circ} \text{ So } B \text{ is preopen}$$

$$\because \bar{B} = \bar{B}^{\circ} \text{ So } \bar{B} \text{ is open}$$

The following is an example for 4-interior closure

Example 9:

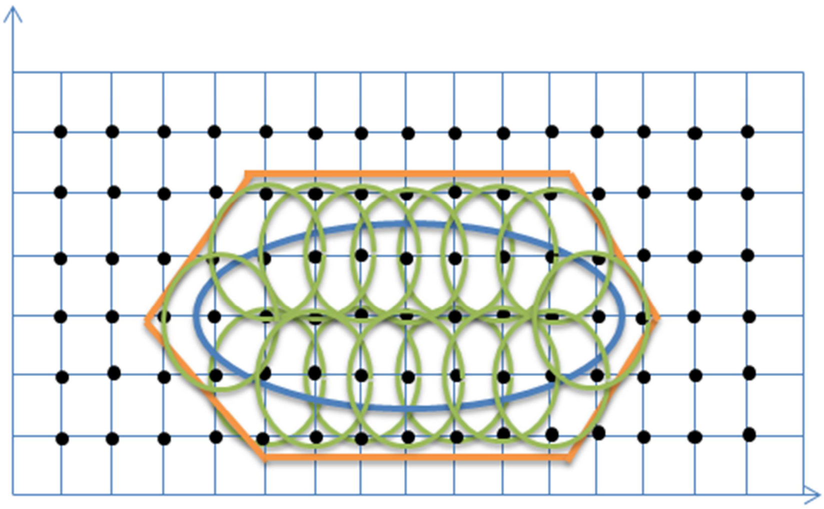

Figure 9. *4-interior closure.*

$\operatorname{int}(\overline{\mathrm{B}}) = \overline{\mathrm{B}}$

Proposition 4:

In general closure operator isn't unitary ($\overline{\overline{B}} \neq \bar{B}$).

The following is an example to show that ($\overline{\overline{B}} \neq \bar{B}$).

Example 10:

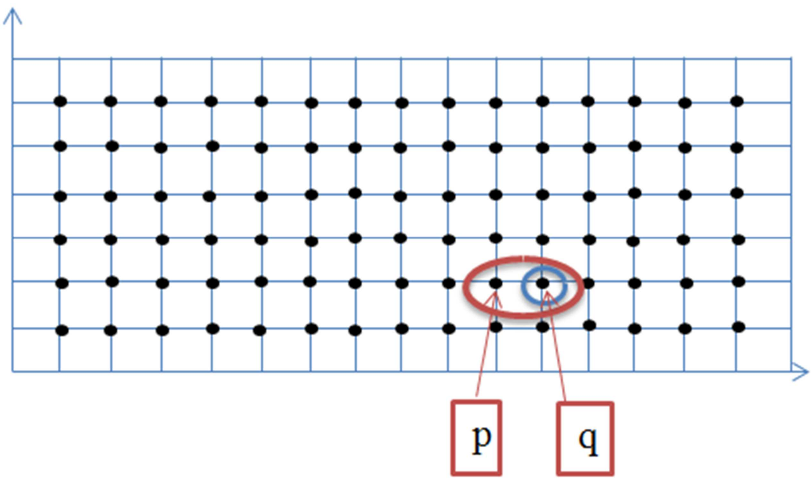

Figure 10. *Shows that ($\overline{\overline{B}} \neq \bar{B}$).*

- Set (B)
- 4-Supra Closure (B) = ($\bar{B}$).
- $\overline{\overline{B}}$.

Definition 5:

1. X is connected if $x = A \cup B, A, B$ *are disjoint open sets*.

2. If $(X, \tau) is\ a\ space$ and $A \subset X, A\ is\ connected\ if\ (A, \tau_A) is\ connected.$

3. wo points p, q will be connected if $\tau_{\{p,q\}}$ is connected.

The following is an example for connected points

Example 11:

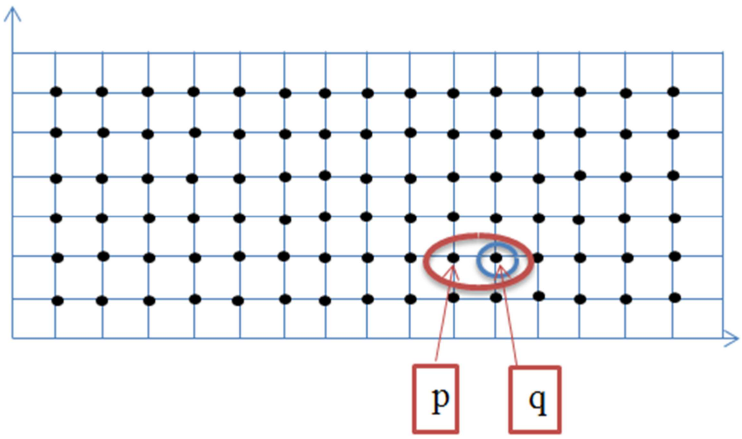

Figure 11. *Connected points.*

$$X = \{p, q\}$$

$$\tau = \{X, \phi, \{q\}\}$$

Since τ contains $\{q\}$ only.

The following is an example for not connected points

Example 12:

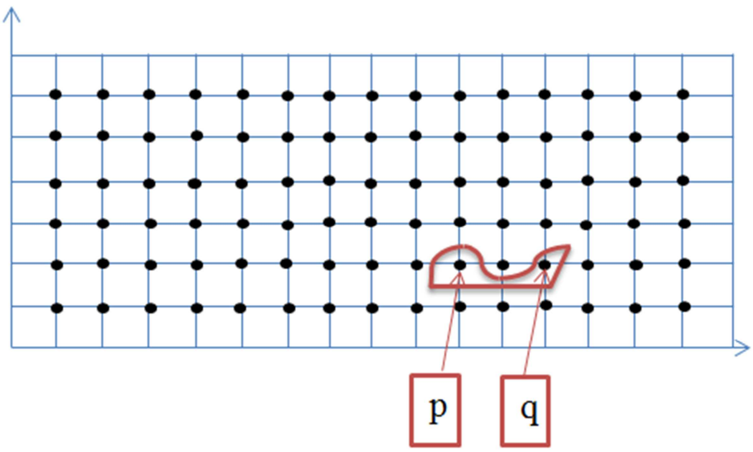

Figure 12. *Not connected points.*

$$X = \{p, q\}$$

$$\tau = \{X, \phi, \{p\}, \{q\}\} \text{, since } \tau \text{ contains } \{p\}, \{q\}.$$

Conclusions

Supra topological structures is initiated in this work, is an attempt to solve connectivity paradoxes resulted from 4 and 8 connectivity. This is to the last of our knowledge is the first construction to solve the problem via topological structures. ■

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